THE
MATHEMATICAL GAZETTE

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THOMAS ARCHER HIRST.

THOMAS ARCHER HIRST, first President of the A.I.G.T., was born in 1830. Articled to a land agent and surveyor at Halifax, he there met Tyndall, with whom he formed a lasting friendship; it was doubtless this friendship which caused Hirst to follow Tyndall to Marburg, where Hirst took a Ph.D. in mathematics in 1852. A brief visit to Göttingen and a meeting with Gauss followed. For a few years he lectured at Queenwood College, Hampshire, but then went abroad again; in Paris he met Chasles and in Rome Cremona. Hirst returned to London first as Professor of Physics and then as Professor of Pure Mathematics at University College; later he became Assistant Registrar of London University. In 1873 he was appointed the first Director of Studies of the newly-established Royal Naval College, Greenwich, an appointment which he relinquished on grounds of ill-health in 1883.

Hirst's main interest was in the study of geometrical transformations, and his work, while closely linked to continental researches, was at the time little understood or appreciated in this country. His reputation abroad stood fairly high, as can be inferred from the following extract from a letter written by Mannheim on the occasion of Hirst's death in 1892, quoted in the Annual Reports of the Association: "M. Chasles appreciait beaucoup Hirst, et . . . il l'a souvent cité dans son ouvrage sur les Progrès de la Géométrie en France. C'était une distinction, car il n'aimait pas à parler des savants étrangers."

Hirst presided at the first eight annual meetings of the Association, and from the Annual Reports it is clear that he took a deep and active interest in the work of the Association. His Presidential Addresses and his numerous contributions to discussions show him actively participating in the efforts that were being made to substitute a more suitable discipline than that of Euclid in the education of young children. "Freedom for the teacher and student, accuracy in teaching and study, Dr. Hirst regarded as essential conditions of the improvements he advocated."

The plate is from a photograph in the office of the Dean of the Royal Naval College. Thanks are due to the present Dean, Instructor Captain A. F. Akhúrst, R.N., and the Photographic Department of the R.N. College for enabling the plate to be prepared and used.

THE PLACE OF MATHEMATICS IN SECONDARY (MODERN) SCHOOLS.*

THE **Chairman** (**Mr. A. W. Siddons**) expressed the view that members were about to take part in the most important of the discussions during that April meeting. He hoped the outcome would be many more discussions in the Branches on " The Place of Mathematics in Secondary (Modern) Schools ".

Mr. F. J. Swan (Camden Training College) : If it be true that " Fools rush in where angels fear to tread ", my presence here as opener of this discussion is clear proof of the fact that I am no angel.

As an introduction, I cannot do better than to borrow and adapt a few opening remarks from the address given by Dr. G. B. Jeffery in January last to the Association of Teachers in Colleges and Departments of Education. Until recently I occupied a position as mathematical master in a grammar school and, as far as mathematics in secondary modern schools was concerned, I was an onlooker and could find fault or propound remedies with the ease that comes from the comforting knowledge that somebody else would have to do the work. Then, suddenly, I found myself in a new position. In September last I began to take a small part in the training of some of the teachers who will serve in these secondary modern schools. That made me think, and think hard. Perhaps the most useful contribution which I can make to this discussion is to pass on to you some of the random thoughts which have come to me during the last eight months.

It is necessary, at the outset, to consider what has happened since the passing of the 1944 Act. No fairy wand has been waved by the Ministry of Education, no new buildings have appeared, no magician has restored the youth of war-weary teachers, and there are no signs that our pupils have become less unteachably unintelligent. In spite of all, there seems to be a freshening breeze, and on the horizon a cloud as yet no bigger than a man's hand. True, there are some in the grammar schools who fear that the cloud may develop and bring storm and devastation, but many in the secondary modern schools look for refreshing showers.

There will, as a result of the Act, be a change in the distribution of children to the various secondary schools controlled by the Local Education Authorities. After passing through a junior school, boys and girls will, at the age of 11+, proceed to a school of grammar, technical or modern type. The choice will be based partly on the expressed wishes of the parents, partly on the pupils' school records, and partly on the results of attainment and intelligence tests. As it is probable that for a time at least the results of the tests will outweigh the other considerations, it is important to consider the likely distribution from the point of view of these tests. If we set the attainments out in a normal frequency distribution we find that the 25 per cent. of the pupils who will probably go to the grammar and technical schools will have I.Q.'s of 110 or over (100 is the mean, median or mode). Thus, of the pupils assigned to the secondary modern schools only one-third will be of average intelligence or above and one-sixth will have I.Q.'s of 84 or less. It is necessary to note that even with an I.Q. of 116 a pupil will probably be in the "B" or "C" stream of a grammar or technical school. It is admitted that at present the psychologists have had little or no success in estimating emotional factors, but a child with a low I.Q. is, without doubt, severely handicapped.

The problem of the mathematics teacher in the secondary modern school is, therefore, entirely different from that of the teacher in the grammar or

* A discussion at the Annual Meeting of the Mathematical Association, April 25, 1946.

technical school. The pupils in the secondary modern schools cover a wider range of intelligence and, what is even more significant, a range at the lower end of the scale. I want to put this point with all the force and sincerity at my command. It is useless to try to force on these modern schools a curriculum of the grammar school type. It just will not work. Children of lower intelligence do not take easily to ideas, for them words are no substitute for realities. We must remember that at the lower end of the scale the pupils will be little more intelligent than those who in special schools learn to do simple routine jobs only after long practice and much repetition. Do not misunderstand me; this is no defence of mechanical mathematics mechanically taught.

Sir John Adams in his book, *Errors in School*, suggests that a distinction should be made between the *static idea* and the *dynamic idea*. For the first he suggests leaving the term "*idea*", and for the second he prefers to use the term "*concept*". He urges, too, that we should leave the "*idea*" to the tender mercies of the "stay-putters" and that we come to grips with the dynamic "*concept*". May I read a short passage from this book: "The old-fashioned folks were very fond of defining, and were apt to say that unless a person could define a term he was not master of the idea it represented. But experience shows us that comparatively few people can define logically the terms they use, and yet they use them correctly enough. The newer way of estimating knowledge of a term is to gauge how far a person can behave intelligently to the matter represented by the term. A man may be quite unable to give a logical definition of a radio-set, and yet may be able to behave himself quite intelligently in relation to such a set." Need I point the moral to such a learned assembly?

In less than twelve months from now the raising of the school leaving age will be an accomplished fact. We are promised or, if you will, threatened with multilateral schools and with plans for a common scheme for all pupils up to the age of 13+. In any case there is a possibility that pupils will be transferred from one type of course or school to another at any age between 11+ and 16. Whether we will or no, many of us will be affected by these changes, and we must be vitally concerned with the new schemes of education. Perhaps I may be allowed in a few words to outline the proposed London plan for multilateral or comprehensive schools? Two types of school are planned, one of 2,500 pupils with a fifteen form entry, and one of 1,500 pupils with a ten form entry. It is not within my province this afternoon to express an opinion on the advisability of these enormous schools, nor to discuss the political and other forces which have played a part in bringing the plans into being. As teachers we are concerned with helping to formulate an integrated curriculum, and more especially as teachers of mathematics our job will be to devise an integrated mathematics course which will be an essential part of the main integrated curriculum. If we do not begin to face this issue at once with the strictest intellectual integrity, there is a grave danger that a few more scraps will be added to our present schemes of work which too often consist largely of unrelated bits and pieces, and in no subject is there greater danger than in our own subject of mathematics. The tragedy is that once any scrap finds its way into our schemes we find excuses which we dignify by the name of reasons for keeping the scrap however useless it may prove.

I intend deliberately to gloss over the general question of control of these new schools. The methods of control will be dictated largely from above (or below, according to your point of view), and we are not really concerned at the moment with this matter. It is, however, necessary to say a few words about staff and equipment. At the moment most of the teachers in the modern schools are general practitioners, and there are comparatively few

specialist teachers in these schools except perhaps in handicraft, domestic science, and less frequently, in art and P.T. There is no time to discuss the merits and demerits of this system, but one thing is clear, either there must be an increase in the number of specialist teachers in these schools or specialists from outside must help in working out the new curricula. The standard of equipment in the secondary modern schools is low on an average compared with that of the secondary grammar schools, except, perhaps, that more use is made of radio and films in the modern schools. The provision of suitable equipment is a most important consideration, and perhaps our much abused public schools may serve as a pattern and, if so, they will have done something to justify themselves.

And now to come to the crux of the matter. In order to determine the place of mathematics we must briefly survey the whole educational field. What is our conception of the true purpose of education? On our answer to this general question will depend the answer to the more limited question which we are here to discuss. For your comfort I say at once that I have no new definition of education. Our educational system must, of necessity, be largely a reflection of our social system. Yet, remembering the awful example of Nazi Germany, we admit that education properly directed can do much to help in bringing about any desired change in the social system. A word of warning is necessary here; the motive force for the educational drive in Germany came primarily not from the educationists but from the politicians. If my interpretation of the history of English education is correct, the purpose of education in the past has been mainly utilitarian. In the Middle Ages, when education was limited to scholars and clerks, the curriculum was scholarly. At the time of the Industrial Revolution education was planned to meet the minimum demands of the workers in the satanic mills and factories. At the beginning of this century there was something of a reaction and we talked of education for leisure, education for citizenship and the revival of the arts. To-day, the pendulum is swinging back and there is a call for more skilled workers, and consequently for more facilities for the study of technical subjects. I make no apology for devoting so much of my opening talk to general topics, for our conception of the purpose of education will determine, among other things, the substance of the curriculum and the methods of presentation.

One could talk for hours on the purpose of education, but I would stress just one point. No philosophy of education can be truly purposeful unless it is child-centred—we must put the child in our midst—we must provide an integrated curriculum for a whole child. There has been far too much adult analysis in this matter of education, and it is high time we had some child synthesis. Children are busy people when engaged on their own lawful business, and it is our job to find out the child's line of business and to refrain from pressing him into our adult business.

May I conclude this general part of my talk by quoting from H. C. Dent's *A New Order in English Education*.

"The problem of the education of the young is twofold. It is to provide for them an environment, activities, and guidance which shall be appropriate to their age and ability, *and thus afford them a satisfying life in the present*; and which shall at the same time discover, draw out, and develop their potentialities, integrate their capacities, and progressively train them so that they shall be fully equipped to undertake the varied responsibilities of adult life."

We are, I think, bound to admit that in the past we have been more concerned with the preparation for the adult life of our pupil than with the provision of this satisfying life in the present. Indeed, we may well ask if the

provision of the satisfying school life is not, after all, the best preparation for the future.

So far we have been concerned with an analysis of our problem. In view of the complexity of modern society and modern knowledge it is inevitable that there should be analysis. Our school curriculum has been analysed into subjects and our subjects split into smaller pieces. Even our pupils, too troublesome for us as whole human beings, have been divided into bits and pieces by our friends the psychologists. Analysis comes more easily to us than synthesis. From earliest childhood we have delighted in pulling things (and later people) to pieces. "To synthesize", says Jacks in his recently published book, *Total Education*, "is a more formidable task than to analyse, and it can only be accomplished by the choicest spirits."

The task, then, of fitting mathematics into its proper place is one which will call for a great joint effort by all the wisest and most skilled teachers of mathematics. The object we have in view will not be gained by discussing this or that clever way of teaching this or that little bit of mathematics, nor will the introduction of snippets of history or the hare and tortoise problems of the old schoolmen solve our problem. We must finish once and for all with bits and pieces and undertake at once and together this vital and difficult task of synthesis.

Perhaps the best pattern of things to come is to be found in our kindergarten schools. In these schools an attempt has been made to provide young children with an environment, activities, and guidance appropriate to their age and ability, and the youngsters certainly seem to have a satisfying life in the present. I would recommend all teachers of mathematics to read the books of Monteith and Drummond.

It is clearly impossible to lay down a satisfactory scheme for our secondary schools until our ideas have been subjected to test and experiment. This period of experiment may well cover a period of years and there must be an interim plan. Little harm can come from temporary expedients if we are determined to discard all that is proved to be worthless in the interim schemes.

So far very little experimental work has been done on the subject of the curriculum, and from time to time I have jotted down some topics which I think need investigation.

First and foremost it is necessary to determine as accurately as may be what mathematics is appropriate to each age and ability. If what we teach at any stage is inappropriate, then our material clearly cannot find a place in any integrated scheme of education. Many teachers in secondary grammar schools know how much has been gained by the postponement of the introduction of formal geometry—perhaps it would be better to postpone it until after the school certificate stage. But I must avoid hornets' nests and red herrings. Here are a few of the questions that will arise: Should decimals be taught before fractions? Should volume come before area? When should percentages be taught? What is the best mental age at which to introduce algebraic symbolism? How early in the course may logarithms be taught? These, and hundreds of similar questions, cannot and must not be answered by a mere expression of opinion. We have suffered too much in the past from opinions not based on experience or experiment. Some experimental work along these lines has been attempted in America and a little in this country, but the total amount to date is small.

Again, there is a wide field for investigation into the methods best suited for our pupils. The publications of the Scottish Council for Research in Education contain some valuable statistics. It is clearly undesirable and unwise to attempt too much standardisation of method. On the other hand, it is criminal to befog our pupils by using methods which experiment proves

to be unsuitable or to expect pupils to change methods during their school course. To instance just one example. Every teacher knows the confusion which arises from the use of the three methods of subtraction, namely decomposition, complementary addition and equal additions. It is surely our business to avoid such confusion. It is easily possible to determine which of the three methods is measurably superior in securing accuracy and speed in working examples. Statistics collected by the Scottish Council have led to the recommendation that the decomposition method be totally rejected, but it is not quite certain which of the other two methods is the better. It is useless to talk of the teacher's personality and independence in such matters. The child's happiness and progress must be considered before the teacher's personality, or is it not rather too often the teacher's obstinacy and prejudice? Perhaps we may yet be called on to decide whether the man who invented standard form should be condemned to death or merely sentenced to life imprisonment.

Then, too, there is the vexed question of problems. We feel that our pupils should be thrilled by our problems. The fact is that children of just average intelligence or lower find problems very difficult and therefore dislike them. There is, of course, a language difficulty in problems, and I await with much interest the results of an investigation into the use of problems in mathematical teaching. My own observations lead me to believe that the problems we use must be within the child's experience; indeed, we may have to create the experience before setting the problem. I feel, too, that more use should be made of what Potter calls the " all practical or figureless example "—the example in which the pupil is set to collect the data necessary for the solution of the problem.

There are many other things I wanted to say, but time presses. The subject is so vast that I have only been able to deal with a few aspects of our problem in a very sketchy manner. I have deliberately kept to the more general aspects for two reasons, firstly, because I believe that we cannot proceed with the solution of our problem until we have laid down certain guiding principles; and, secondly, because those who are to follow me are far better qualified to present detailed schemes.

In conclusion, may I say that I believe our job as mathematics teachers is to build up in our pupils first of all a sound number consciousness and later a sound space consciousness, and that all our work should be intimately related to a central theme of education. True, our work is only a small part of the integrated scheme of education, but the whole cannot be perfect if our part is in any way imperfect.

Mr. C. T. Lear Caton (Alcester Grammar School): My first reaction to the title of this discussion was a feeling of doubt as to whether Mathematics had any place at all in the curriculum of secondary (modern) schools, at any rate if the word " Mathematics " connotes the kind of algebra and geometry traditionally taught in grammar schools. I must admit that at the time my experience of modern schools was slight, and I suspect that the same is true of most of my audience this afternoon, but after accepting the invitation to take part in this discussion, I made it my business to visit a number of modern schools in diverse types of district, from small rural schools to large modern schools in urban areas, with the objects of sampling the intellectual ability of the pupils and getting to know the points of view of the head teachers and their staffs.

The distinction usually made is that modern schools contain those pupils who can best learn by practical work and through the concrete, while grammar schools are for those boys and girls who can best learn from books and can appreciate the abstract. If this were the whole truth, the selection of a curri-

culum in mathematics for modern schools would present little difficulty. Mathematics is a subject which happens to have two sides to it : in part it is a subject of evident necessary usefulness ; every citizen of the modern world ought to have a certain amount of mathematical equipment to appreciate intelligently his surroundings and to make certain aspects of his experience more interesting. Other parts of mathematics have no immediate practical value, but form an intellectual pursuit which gives much satisfaction and delight to a certain type of mind. This double aspect is clearly seen in the case of Geometry. As regards its concepts, its use of instruments, its training in the drawing of figures and projections, its education in special conception, it is a practical subject and a necessary element of knowledge for any intelligent person. On the other hand, the Euclid type of geometry, the familiar logical structure of theorems and riders, has little practical value, but it can provide much interest and mental stimulus for the natural mathematician whose mind profits by its study. It is obvious that while we can suitably teach both kinds of mathematics to the academic type of pupil in the grammar school, we should restrict our curriculum to the practical aspect of the subject for the boy or girl in the modern school.

Now we have to ascertain whether the modern school pupil is capable of absorbing very much even of this practical type of mathematics. The first point to be appreciated about modern schools is that by far the largest part of the school population is included within them. If we take the usual distribution curve for intelligence, a narrow strip at the top end, perhaps one-seventh of the whole, accounts for the whole of the children in the grammar schools, a similar narrow strip at the bottom end corresponds to mental deficients and backward children in special schools. The whole of the rest of the area below the curve corresponds to the modern school population.

If we are considering teaching some practical mathematics to the better half only of the modern school population, we are even then planning a syllabus for more children than the whole of the pupils in the grammar schools put together.

The second point to be borne in mind is that the ability of pupils in modern schools varies enormously, both within any one school and also from school to school. In grammar schools the imposition of examinations at the beginning and end of the course keeps the work and abilities of the pupils fairly homogeneous. In a modern school the lowest abilities will only be slightly above the level of retarded children. The upper limit of ability depends on the district ; if there is a plentiful provision of secondary schools, the elementary schools will be well " creamed " by the grammar school intake of pupils ; if secondary school provision is meagre, the modern schools will take in a considerable number of pupils of good ability. You may find that a modern school contains pupils who at the age of 12 cannot adequately use the twice times table, and also pupils who could easily pass a School Certificate mathematics paper after the usual course of training. Clearly we shall not have to provide a single curriculum for the modern school ; we need rather to consider several curricula for the varying " streams " of pupils according to their mathematical ability and degree of intelligence.

I find that these differences are reflected in the opinions of teaching staffs —in some modern schools the feeling is that the children are too dull to do any mathematics other than arithmetic ; in other schools the view is held that the present syllabus is too restricted, and that the better pupils could do far more algebra and geometry than is now expected. I prefer to look at the matter from a quite different point of view. There is a certain amount of mathematical knowledge which any sensible citizen ought to possess. Teaching methods ought then to be found for imparting this mathematical know-

ledge to as many boys and girls as possible, in any type of school, certainly to the majority of children. If three-quarters of our children are to pass through modern schools, the need for providing a curriculum in mathematics and suggestions for means of instruction for modern school pupils becomes obvious. I suggest that it is appropriate for our Association at this time to put forward tentative proposals on these lines for consideration by those responsible for the curricula of the new modern schools.

Now the modern school curriculum cannot be considered altogether apart from the grammar school curriculum. This is especially important in view of the new possibility of transfers of pupils being made from one type of school to the other during the first two years of the course. Many arguments are being adduced both for and against such transfers being made, but it cannot be gainsaid that no selective tests at the age of 11+ can be sufficiently reliable to decide such a vital matter to elementary school pupils as the type of secondary education they are to be offered, in view of the decisive effect on the whole of their subsequent careers. It is not the time to discuss the defects of the secondary grammar schools admissions examination, but I may mention the disadvantage suffered by the late-developer, who is often the more intelligent type of child, and I would draw attention to the increasing tendency to use tests in which the marking is statistically determined, such as Intelligence tests, and English and Arithmetic tests of a similar character, to give results for individual pupils, whereas from the nature of the marking such tests might be expected only to give entirely reliable results when applied to averages over a group of pupils. Such considerations make it inevitable that some form of adjustment by transfer from school to school must be provided. The difficulty of such transfers will be much reduced if for a time (probably the first two years) the modern and grammar schools have a common curriculum. In mathematics this is not so difficult as might be supposed at first sight. As we have said, mathematics in one aspect is a subject of necessary practical utility. If we can agree on a schedule of mathematical work which ought to be familiar to every citizen of the modern world who has the mental capacity to learn it, could we not then teach this work during the first two years when transfers are being made?

The Yorkshire Branch of the Association has already considered in detail proposed schemes of instruction in mathematics for modern, grammar and technical schools, bearing in mind the advisability of a common core during the early years of the course, and Mr. Watts will shortly be speaking to you on this matter.

Such a common core for the first two years would certainly involve some re-arrangement of the mathematical work in grammar schools, for example, the postponement of the beginnings of deductive geometry and manipulative algebra to the third year, but I suggest that such a postponement would be an advantage and would fit in better with the psychological theories of the child's development. Even a bright boy or girl entering a secondary grammar school at the lower age limit of 10+ is not ready for deductive reasoning until the age of 12+, when he will be beginning the third year of the course.

In approach, the modern school pupil requires a practical viewpoint. There has been a tendency in this direction for some time amongst grammar school mathematics teachers, and I suggest that a further move towards a practical approach in the early stages will be beneficial to the grammar school pupils too. It goes without saying that the teaching work in any school should be linked as closely as possible to present-day and historical applications of mathematical methods, in order to give the work a wider educational usefulness, and I suggest further that in this part of the curriculum mathematics should be taught as a single subject, not as several

branches. The later stages of grammar school mathematics do not lend themselves so well to integrated treatment, but I think there is a growing opinion that this is advisable in the initial stages.

It is obvious, of course, that the various streams of pupils could not all progress at the same rate. We have already seen that several curricula will be needed. The weaker modern school pupils could not attempt the whole course at all, but would have to do as much as they were capable of doing. At the other extreme, the more able and mathematical pupils in the grammar school could do more than this course in the time, though in my opinion it would be better for them to devote more time to the humanities, especially if specialising in mathematics later. If we agree on a syllabus for the " average pupil " (*e.g.* the better half of the modern school and the weaker half of the grammar school pupils), deviations for different streams could be provided.

At the end of two years, the work of the two types of school would naturally be differentiated. The readjustments of pupils should have resulted in the grammar schools containing the pupils who appreciated knowledge for its own sake, the modern schools those who valued knowledge for what it could do. In the modern school, the more mathematical pupils could appropriately be brought on to the technical facility they would need in the subject if they wish later to take up work needing mathematics or to enter technical schools. A course in algebraic manipulations and methods, some algebraic trigonometry, an acquaintance with the calculus and coordinate geometry, familiarity with mathematical devices such as logarithms, slide rules, verniers, and nomograms, more advanced geometrical drawing and some treatment of mechanics, provide topics which are clearly useful and could be taught to the better pupils by practical and realistic methods. Work such as deductive geometry which has little applicational value is appropriate for the grammar school pupils only, and is out of place for the modern school scholar.

There is one final point I should like to emphasise. A leading difficulty of modern schools will be found in the fact that few of modern school teachers are specialists in mathematics. The specialist system of grammar schools has its critics, but it has one immense advantage. The subtle but vital feature of good mathematical teaching is somehow to convey to the pupils the mathematician's outlook. If so educated, a pupil on leaving school may quickly forget the details of the mathematical methods taught, but he or she will retain a form of mathematical consciousness which will enable him or her on meeting a mathematical problem in later life to tackle it from the point of view of a mathematician. Such teaching is only possible to the man or woman who has been immersed in a specialist mathematical training. It is to be hoped that the equation of salary scales between types of schools will result in some mathematical specialists being attracted to the modern schools, so as to do the mathematical teaching at least in the upper forms. I think this Association can do much to help by welcoming mathematics teachers in modern schools to its branches, by arranging programmes for them, and most of all by giving them the benefit of contact with mathematical specialists. If this is done, I believe that the specialist teachers in grammar schools will find in their turn that they have something to learn in teaching methods from their modern school colleagues.

Mr. H. H. Watts (West Leeds High School) : I feel a certain trepidation at venturing to speak in this discussion at this stage. I have no experience of teaching in schools other than secondary grammar schools, and my only excuse for my appearance here is that the Yorkshire Branch of the Mathematical Association at my instigation set up in November, 1943, a sub-committee to consider possible syllabuses in mathematics for the proposed three types of secondary school. We held numerous meetings, and teachers in

technical and, as they then were, senior schools gave us their opinions on our proposals, and also offered advice which was a result of their personal experiences. The sub-committee reported back to the Branch at three meetings in 1944, when its proposals were discussed and amended. Finally a Report embodying the suggested syllabuses was published and sent to Branch members, and it is upon this that I wish to base my remarks now.

Before coming to the actual suggestions for the curriculum in mathematics I should, however, like to make a few general comments. In reading articles and reports dealing with mathematics in schools one has often come across statements that mathematics is disliked by children, that they find it dry and uninteresting, and what a pity this is. For example, *The Content of Education* (the Interim Report of the Council for Curriculum Reform), in the chapter dealing with mathematics, says : " Is the unpopularity of mathematics inevitable? for unpopular it certainly is." It goes on to say that this matters, matters enormously, and then to suggest its remedy.

This is not an isolated case, but I think that most practising mathematical teachers would refute the statement. Actually, few children really dislike mathematics, and many of them enjoy doing it—in the elementary stages, at least. It provides them with a definite task, something they do themselves and, provided that exercises are carefully chosen and are well-graded, even the weakest in mathematical ability among them can get something correct and can get that glow of self-satisfaction which follows success.

An interesting light on this subject is given in the story of the *Rugby Day Continuation School*, by P. I. Kitchen, its first headmaster. When the school was opened it was thought that the children must be entertained and amused, and syllabuses were devised accordingly. Naturally, parents and employers grumbled : " Nothing to help them with their jobs ", but to the surprise of the school authorities " the young people themselves found an overdose of sweetmeats indigestible." They did not like mathematical problems they could always solve, and so schemes of work have been replanned with more solid, more difficult and more challenging, if duller, tasks.

Again, the Hadow Report of 1926 quotes statements from heads of central schools saying that the favourite subjects are mathematics, science and drawing. As the modern schools will fulfil a function somewhat analogous to that of the central schools—where these were not " selective "—it is obvious that this remark applies to the sort of pupil who will be found in these schools.

When we come to consider the actual content of the curriculum we are faced with lack of knowledge of what is possible. Let us look briefly at the historical development of the modern school. The growth of the educational system in England with the cleavage between primary and secondary education, the fact that elementary schools have always been schools for the poor, giving an education of a minimum standard, has meant that efforts to do more have always been hindered by poverty, by large classes, inadequate textbooks, poor buildings, etc. The first attempt to give the children something better was the establishment of higher grade schools between 1880 and 1900. These dealt with selected pupils, and tried to give an education akin to secondary but with a practical bias ; thus we find mensuration, geometrical drawing, machine drawing, practical geometry featuring in the mathematical syllabuses. Most of these schools became secondary schools after the passing of the 1902 Education Act.

The " Fisher " Act came in 1918, and we find its effects in the so-called advanced courses in elementary schools and in the idea of day continuation schools. Leeds had advanced courses of three years' duration in 27 boys' schools and 11 girls' schools with syllabuses adapted for Leeds needs, that is, with technical, industrial and commercial bias—which meant largely that

geometrical drawing and machine drawing were introduced as additional subjects into the curriculum.

Next came the Hadow Report of 1926. Its terms of reference were, *inter alia*, to consider the content of courses of study suitable for children who will remain in schools—other than secondary schools—until 15. It proposed the break at 11+, and suggested the name "Modern School" for schools giving a course with a "realistic" or practical trend in the last two years. The syllabus suggested includes algebra, trigonometry, practical geometry and logarithms. While admitting that the actual content of the syllabus will depend upon the individual school to some extent, it states that it is very desirable that the courses in all schools should be approximately the same. Further, "for the first two years the work will be mainly fundamental and will not vary materially whether in Grammar or Modern school, and in the better Modern school the course will be comparable with that in the Grammar school—if the foundations laid in the first two years are similar transference to the Grammar school will be facilitated".

It is difficult to discover how far these suggestions have proved effective. Some areas re-organised quickly, others have not done so yet. I know that in some schools work of a very valuable kind has been done, but in others there has been little change. In discussions with teachers the reasons advanced are usually the same: big classes, no equipment, poor textbooks. There has been a move towards "Civic" arithmetic; some schools have introduced such topics as surveying and have done simple geometry and elementary algebra, but the mathematics has remained largely arithmetic.

Certainly the work done in some Evening Institutes to which the children pass on leaving school at 14 bears this out. The syllabuses of one such school which sets out to prepare elementary school-leavers for National Certificate courses in two years' time are trivial: fractions, mensuration, use of symbols, factors, graphs, angles and their measurement, facts about circles!

As I have said, my experience is limited, and I am sure that much more is being done in schools in which the teachers are really interested in mathematics. But the levels of attainment are low in many cases; the abilities of children in a large class vary enormously, and this makes progress to a good standard difficult.

The question of the extent to which a "bias" towards after-school occupations should be given has always been a prominent one in the consideration of the curriculum of the non-academic type of school. Thus we find the higher grade schools and the advanced courses in senior schools giving instruction in machine drawing and commercial arithmetic. However, the Hadow Report states: "Bias should not be of so marked a character as to prejudice the general education of the pupil." Numerous committees set up by individual industries report similarly. They stress the importance of a general education. No special qualifications are needed in most industries, "a readiness to think things out", "receptivity of mind", "adaptability", are the qualities demanded. Several large distributing firms told the Hadow Committee that a thorough knowledge of elementary mathematics was especially valuable.

From these considerations three facts seem to emerge:

(i) Some mathematics must find a place in any school curriculum.

(ii) The actual content of the syllabus in the modern school is likely to be in doubt for many years; past experience is not a reliable guide to what might be done in the future. With smaller classes, better-trained teachers and good equipment the standard of work will inevitably be raised. How much, only time will show.

(iii) The same standard of work cannot be expected of all schools and of

all children. In the secondary grammar schools we meet great differences of ability; how much more so will they be found in the secondary modern schools which will contain five-sixths, and not one-sixth, of our children.

When we come to consider the actual courses in the modern schools I think we must agree that those of the first two years should be similar to those of the grammar and technical schools so as to allow for transfer at 13+. How much transfer will occur is doubtful, but that it should be planned for is indisputable. Granting this then, what should the syllabus contain? The differences in the abilities of children and in their attainments in two years are enormous, and it would be impossible to lay down a definite course and to say that all pupils must cover it, neither more nor less. Obviously, much will depend on staffing and internal organisation, but under good conditions much improvement is possible, and, as only the "A" stream pupils in the modern school are likely to transfer, it should be possible for them to work to a syllabus similar to that in the grammar and technical schools. Thus the Yorkshire Branch committee to which I have referred—and I notice that the Lancashire Branch committee a year later reported similarly—suggests syllabuses which are the same for all schools, but give additional matter which the grammar schools may include if they think it desirable. The modern school pupil who is transferred will be of sufficient ability to catch up.

The Yorkshire Branch committee discussed their syllabus with teachers, head and assistant, from senior elementary schools, and it was adjusted until these teachers agreed that it was within the reach of their better pupils. Here is the proposed syllabus:

SYLLABUS FOR THE FIRST TWO YEARS IN ALL SECONDARY SCHOOLS.

ARITHMETIC.—The pupils entering at 11+ are expected to know their tables and the four rules with simple applications. The aim of the course is to acquire elementary technique and manipulative skill.

General revision. Introduction to vulgar and decimal fractions. Simple practice. Averages. Factors.
The four rules applied to vulgar and decimal fractions.
Simple facts of the metric system.
Unitary method.
Plotting of simple graphs from tabulated values.
Decimalisation of money, weights and measures.
Square root. Ratio and proportion.
Percentages. Easy profit and loss. Simple interest.
Simple areas of rectangles and volumes of rectangular solids. Mensuration of the circle ($\pi = 22/7$).

ALGEBRA.—The aim is to obtain facility in the use of symbols in arithmetical processes.

Literal expression. Formulae. Substitution in formulae.
Simple equations and problems.
The four rules applied to positive and negative numbers.
Multiplication and division by single terms.
Use of brackets.
Common factors. Difference of two squares.
Elementary operations with fractions having monomial denominators.
Plotting from simple formulae.

GEOMETRY.—The aim is to give the pupil: (*a*) a background of geometrical knowledge; (*b*) facility in the use of compasses, set squares and protractor; (*c*) an appreciation of geometrical argument.

The method of approach is important. The facts should be discovered

experimentally and then established by discussion and reasoning. They should be used in a large variety of problems, both numerical and theoretical. No formal writing out of proofs should be expected, but geometrical argument should be included. Freehand figures should be encouraged where appropriate.

Solids, surfaces, lines and points.
Measurement of angles. Angles at a point.
Angles formed by parallel lines.
Angles of a triangle and of polygons.
Construction of triangles. Scale drawing in height and distance problems.
Congruent triangles. Idea of similarity.
Construction of quadrilaterals.
Problems involving simple loci.
Ruler and compass constructions for bisecting lines and angles and for drawing perpendiculars and parallels.
Practical acquaintance with the common geometrical solids.
Applications of congruent triangles to parallelograms, rectangles, the rhombus, with ruler and compass constructions.
Areas of parallelograms and triangles.
Numerical and theoretical applications of Pythagoras' Theorem.
Easy theorems on chords and on angles at the centre and circumference of a circle.

It is not intended that the order given must be maintained; the individual teacher will develop the work as he pleases. Again, it should be emphasised that not all children should be expected to cover all that is suggested but this seemed to be the minimum which could be taken as a "common core" if secondary (grammar) school standards are not to fall.

The fact that this is a "common core" and that a child may pass from one school to another raises the question of teaching methods. It will obviously be an advantage if similar methods can be used in the schools in, at least, one area, but the extent to which this is possible will depend upon the amount of cooperation of the teachers.

When the Yorkshire Branch committee came to consider the syllabus for the next two (later three) years, it was felt that nothing so definite could be laid down. It is here that the question of "bias" will be decided, and the extent of the syllabus will be governed by the resources and environment of the individual school. In fact, in many schools, apart from work needed for other subjects such as science and geography, little more formal work is likely to be attempted. The topics of the first two years will be revised and consolidated, with examples which exhibit its applications to everyday problems and affairs. The course need be no collection of "bits and pieces", but can be an ordered whole. Practical work will have a prominent place, not only in drawing and model-making, but in the working out of many topics.

Thus new work might include logarithms and trigonometry, more mensuration, fuller discussion of similarity, use and construction of formulae, graphs—perhaps allied with some simple discussion of statistics.

Otherwise, there would be "Citizenship" arithmetic for all: national and local finances, rates and taxes, wages, use of a bank, insurance, investments, stocks and shares, companies, balance-sheets, and so on. Then, in industrial areas, problems dealing with workshop practice and industrial transactions would be considered, and geometrical and machine drawing would undoubtedly find a place. In rural areas there is considerable opportunity in connection with farming and agriculture: crop yields, fertilisers, seed requirements, values of stocks, appreciation and depreciation, financial dealings such as loans, interest and subsidies, design of buildings and a host of problems connected with this.

In addition, I suggest that no mathematical syllabus should be considered complete if it does not deal with some of the history of mathematics, taken not necessarily formally—although occasional lessons, particularly if an episcope or projector is available, will be of value and interest—but woven into the lessons so that the children may realise how the subject had developed and the part it has played in the gaining of man's power over nature.

Finally, some simple astronomy would be of interest and of lasting benefit to some of the children and afford interesting applications of the work done in the class-room. If a room can be set apart as a mathematical laboratory and library, then these can be built up and an opportunity given to the children to make their personal contributions.

There is plenty of scope, all that is needed are teachers who know their subject and are prepared to think about it. The modern schools are free from external examinations and free to plan whatever courses they deem suitable, and the Mathematical Association will, I am sure, help in any way it can. May they make the most of their opportunity.

The **Chairman** said that his experience had been mostly in public schools; he had inspected schools and watched the teaching in many others. There was much he could say on the subject but it seemed best that those with practical experience should be the speakers. He wished to say, however, that far more important than the plotting of graphs, so far as the syllabus was concerned, was the reading and interpretation of graphs. When pupils had plotted graphs, they should be given those graphs to read. It was astonishing how interesting that could be to quite young pupils. The Chairman thought Mr. Watts agreed with him as to that, but his skeleton syllabus had not mentioned that point.

Mr. H. H. Watts indicated his agreement.

Mr. E. J. P. Smith (Slough Grammar School) put in a plea that mathematics be regarded as one and indivisible. Now it seemed that those concerned were back as in the days of Old Gaul—divided into three parts. He had been trying for many years past to get into the minds of junior masters that mathematics should be taught as one subject. Nevertheless, only a few years ago he had told a junior that he wanted a paper set in mathematics, not algebra, arithmetic and geometry. There were to be nine questions, and the reply was: "I suppose you want me to give three questions on each?" That was what one was up against all the time. Unification would be much more easy if geometry were made less deductive and not even so much practical as based largely on calculations. With calculation one could put together, straight away, geometry and arithmetic. If teaching fractions, why have them in two places in the syllabus? Fractions were fractions whether x and y were denominators, or 10 and 15; it was the same piece of work, so let it be one job and not two. The syllabus, so far as the work was concerned, was admirable, but there was necessity for combination into one instead of division into three parts; and this should apply not merely to the first two years, but to the whole of the syllabus for the modern schools. There should be an integrated system of mathematics in which there should be one stream of thought running right through. That, of course, would be difficult, but it should be the aim.

The speaker added that he had been glad to hear the suggestion that perhaps intelligence tests were of dubious value. The experience of his school was certainly against them. When he could be assured that the same kind of work was being put into the statistics of the intelligence tests as was being put into the work at Rothamsted Experimental Station he would begin to believe in them, and not until then.

Mr. W. F. Bushell (Birkenhead School) felt a note of gratitude was due to

the three openers for the way they had presented a matter of the very highest importance. The Chairman had commenced by expressing his conviction that members were about to take part in the most important discussion of all during that April session, and when subsequent speakers had stated that the mathematics they were discussing was for some six-sevenths of the population, then all must have entirely agreed with what Mr. Siddons had said.

The speaker next emphasised, as he saw it as an old member of the Mathematical Association, the responsibility of the Association as to what was going to happen. He noted that its membership was quoted in the Report at something like 1775, and he imagined that there were not a great number of those who were teachers in the new modern secondary schools; the great mass of members of the Association were graduates, and the meeting had been told that the new teachers of mathematics in the modern secondary schools were "general practitioners". The history of the Association over the last forty years, in particular since 1903, when mathematical teaching was largely reformed, due to the inspiration of that Association, not only over all England but over a large part of the world—and Mr. Bushell said he was speaking more particularly of the British Dominions—revealed that what the Association had achieved was of a very high order. Now it had something else before it, namely to try to develop the teaching in the modern secondary schools on similar orderly, good and useful lines. He most earnestly hoped that the Association would in no way abdicate the position which it had won for itself after forty years of effort.

Mr. Bushell added that if he was right in believing that the great mass of those "general practitioners" who taught mathematics in the modern secondary schools were not members of the Association then it was the duty of all the present members to try to get them in. That could, he imagined, be most easily done by the Branches, of which there were, he thought, at least 10 in the big centres of population throughout the country. It might be especially possible for Branches in large towns such as Birmingham, Manchester, Liverpool and elsewhere to do something in that respect. He rejoiced to know that in at least one or two cases efforts in that direction had already been made, and on that those concerned deserved congratulations. In Merseyside—his particular area—Mr. Bushell feared that no particular effort had yet been made; he felt he would have to play his part in trying to rectify that. At any rate, there should be an endeavour to draw those "general practitioners" into the Association in the first case as associates, because it would be unwise to frighten them with some of the higher mathematics we saw in the *Gazette*; let them come in as associates, with the aid of local Directors of Education and the like, so that they could enjoy something of what might be the inspiration of an old Association, and the help of the mathematicians in their particular area.

There was another point which needed bringing to the fore. More than once there had been reference to the difference between concrete and abstract mathematics. Mr. Bushell confessed that he wanted to know a little more about that. So often it had been stated that unless those concerned put their house in order, and unless they associated mathematics more with modern everyday life, somebody else would do it for them. He ventured to suggest that the Mathematical Association and its members already tried their utmost to associate mathematics with modern everyday life. It was extraordinarily difficult to make mathematics as concrete as all would desire, but many members of the Association and a great many textbooks did their utmost for concrete mathematics, and very difficult it often was.

Lastly, Mr. Bushell wanted to know something about what the speaker had

called " the common core " of mathematics for boys of 11–13. He took it to mean that it implied the kind of syllabus that had been used for elementary work in grammar schools in time past, and he assumed that it was intended that it should be presented very differently to those who had different intelligence quotients. All knew how teachers first studied the syllabus of the School Certificate, Higher School Certificate, Scholarship, or whatever it might be, but then all looked to see the type of question. The point to determine was the type of question that developed from a particular syllabus, and the speaker imagined that the questions for boys in secondary (modern) schools might well be different from those to which they were accustomed in grammar schools, even if the syllabuses were the same. Granted that, he found no fault with the syllabuses presented for boys of 11–13.

In conclusion, Mr. Bushell reiterated his gratitude to those who had brought forward so admirable and excellent a series of papers, and again he hoped most earnestly that all present would determine to do their utmost to play their part in regard to the mathematics for the secondary modern schools as their predecessors in the Association had played their part for the mathematics in the grammar schools in the past.

Mr. T. A. Grocock (Ripon Grammar School) referred to the methods to be employed in doing what had been urged. In his area there had been meetings of teachers in the various types of schools, and discussion with a view to deciding on the methods to be adopted in doing the various things covered by the syllabus put forward by Mr. Watts. He did not know whether other members had taken part in a similar gathering, but to find any agreement was almost impossible—worse than in the case of the United Nations Organisation! Whatever was brought forward, the argument was that some of the children in the modern schools were too stupid even for that. Nevertheless, a good deal of value had been derived from the discussions. It seemed more important to conduct tests on the teachers than on the pupils, in the first place at any rate. The speaker asked, if there was to be any report on the matter, that the Mathematical Association should put this at the top, that the Association should advise the teachers to get together first and decide on the methods to be used for the very simple things. Then perhaps the children would not be so confused and one could start on them.

Mrs. E. M. Williams felt there was need for a clearer understanding with regard to the supply of teachers in modern schools. Definitely there were not wanted, nor could there be, specialist teachers of mathematics to cope with all the mathematical teaching in those schools. The supply was not sufficient. In addition, it was obvious that in the modern school mathematics would not always be taken as a separate subject in its own right, but it should be part of the study of the child's environment, one of the keys by which the child opened the door to inspiration and understanding. Therefore, it was necessary to look to others besides mathematicians for the teaching of mathematics. It was generally known that a large number of graduates from various universities did, in fact, take posts in modern schools, and it was largely from the ranks of those graduates that the essential specialists for some types of modern schools would be found. Training colleges also provided mathematicians of a kind, and Mrs. Williams wished to speak with respect of those mathematicians who came up with wide human interests but with a capacity for mathematics and an interest in the subject which made them adopt it as their one subject of advanced study during their two-years' course. There were others who took mathematics as just an ordinary subject, and they, too, often offered mathematics but as one teaching subject among others. That seemed to be the pattern of most of the teaching which would be done in modern schools ; people would come into those schools offering

several teaching subjects, of which mathematics might be one. It was clear that the quality of the teaching would, therefore, depend largely on the previous experience of those teachers in their grammar schools. It was for that reason that Mrs. Williams supported the idea that the first two years of mathematics in all types of secondary school should be similar, so that the pupils who had had the same type of two-year practical training in the grammar school could understand the approach required when they went into a modern school.

In regard to the approach, Mrs. Williams doubted whether she ought to say it, but she confessed to being quite horrified by the syllabus put forward by Mr. Watts. It was just the type of syllabus she had drawn up thirty years ago when she started teaching in a grammar school. Her contacts with modern schools had forced her to move away from it. She wished to see something absolutely different in conception. She would like to see mathematics taken at that stage as descriptive, leading to understanding both of number and, concurrently, of space. If that could be done, the children would come on at 13 for specialised work of more technical or abstract kinds, with a sound foundation of practical experience and of ideas, able quickly to acquire techniques. For example, beginning with the idea of drawing to scale, ratio, proportion, mensuration and simple trigonometry would be developed. On the other side, she would have the idea of functionality developing through arithmetic and graphs and involving algebra. In this way mathematical understanding would be won in the study of actual figures and situations.

Mr. S. Inman (Isleworth County School) thought there had been somewhat conflicting suggestions by the opening speakers, though he did not wish that to be taken as indicating that he was not grateful for what they had said. Mr. Swan had stated—and the speaker thoroughly agreed with him—that the problem should be considered from the start as a problem of what to teach in the modern schools. Later speakers had suggested that the curriculum of the modern school should be coordinated with the first two years of the grammar schools ; another speaker had said that there were various curricula and had later suggested the possibility of a common curriculum, and he thought that might be done by postponing all treatment of geometry and algebra for two years. That, thought Mr. Inman, could be done only by sacrifice of the ability and the time of pupils in grammar schools. This the speaker was heartily opposed to. He thought it would be well to get back to the suggestion made by the first speaker and consider what were the requirements of the modern school as such without regard to correlation at the sacrifice of the time and abilities of pupils in grammar schools. That was the first point the speaker wished to emphasise.

He also wished to draw the attention of members present to a report published in the official journal of the I.A.A.M., the *A.M.A.* A questionnaire had been sent to about 400 soldiers who were almost entirely ex-elementary school scholars. Each soldier was asked to name among the subjects which were taught him the two he considered he liked best at school and the two he detested most. Arithmetic came out as the subject which had been most popular with those scholars, and that was rather interesting. Therefore, in tackling the subject which that meeting was discussing, it would be well to avoid the pessimistic point of view ; it seemed that arithmetic had something which, on the whole, pupils took to and liked and, further, if the problem was tackled experimentally, if the best methods and the best topics were gone into experimentally, and so on, it would be possible to achieve something both interesting and of educational value.

Miss Meetham (Abbotsholme) spoke as a result of some experience of

teaching boys whose intelligence quotient was about 100. The syllabus which had been suggested would not fit their needs, and the speaker agreed with Mrs. Williams that the topics which were included should be integrated. Both the arithmetic and geometry of proportion, for instance, could be done at the same time. The speaker would like to see the manipulative algebra greatly reduced. Much mathematics could be combined with general projects on world affairs. She had taken one or two mathematical projects which had worked reasonably well. Surveying was successful for not very clever boys; descriptive astronomy also. Once she had had a timber project, the boys working out the volumes of the trees and the costs of them. She had also tried taking the boys on visits, for example, to the signal box on the railway. It was possible to be too academic, which was bad not only for the less intelligent boys, but it also cheated the clever boys, even the real mathematicians, out of something the school could give them, something which was very valuable.

Mr. H. H. F. Miller (Watford Grammar School) spoke with regard to the transfer of pupils from modern schools to grammar schools at about the age of thirteen. Until about two years ago he had been teaching in a mixed school of about 450 pupils. There was a three-form entry composed of about 60 county minor scholars, a few local scholars, and the balance was made up, in those days, of fee payers. In addition, some 15 to 20 pupils of about the age of thirteen were transferred each year from the modern schools of the area and placed in the Second Form. It will be realised that these pupils formed a substantial proportion of the total numbers in the school. In other subjects no great difficulties were encountered, but French and Mathematics caused some trouble, as the children had missed a year's work in the language as well as in Algebra and Geometry. By the sacrifice of the children's preparation periods and some free periods on the part of the staff, however, they were brought up to standard in these subjects and, by the time they entered the Third Form, they were able on the whole to hold their own very well. It was necessary that they should enter the Second Form, since they would otherwise have been about eighteen by the time they reached School Certificate. There were, of course, other good reasons for not placing the transfer scholars with children two years their juniors.

Pupils were entered for the transfer scholarship examination on the recommendation of their headmasters or headmistresses or at the request of their parents. The speaker regarded this last provision as a necessary safeguard against any possible accusation, by parents or children, that they had not had a fair deal. The examination was conducted by the grammar school staff and consisted of straightforward attainment tests in English and Arithmetic. There was no intelligence test, though the "comprehension" question in English and the problems in Arithmetic were designed to pick out the more promising pupils. The results achieved by the pupils so selected compared very favourably with those of the county minor scholars admitted at the age of eleven, although these latter took a five-year course as against the four years the transfer scholars had in the grammar school before School Certificate.

The speaker wished to emphasise that, in his experience, it was by no means only the mediocre, borderline pupils who failed at 11+. In fact, four of the better mathematicians he taught in that school all failed in the county minor examination. One entered on transfer, proceeded to win a county major scholarship, took a creditable degree in mathematics, and spent the war years on aircraft research work. The others came in as fee-payers, a course no longer possible, and two of them reached county major standard. The speaker was further convinced, from his experience with boys he had met in his A.T.C.

work, that the number of useful pupils who were unsuccessful at 11+ was considerably larger than was generally supposed.

It was, in the speaker's opinion, becoming increasingly recognised that a finally satisfactory selection was not possible at 11+. He did not for a moment imagine that postponement to the age of 13+, a course which it seemed to be becoming increasingly fashionable to recommend, would lead to a selection any more satisfactory, and, since it must necessarily retard the better pupils, it could serve only to lower grammar school standards. To the speaker, the proper course seemed to be to continue selection at the age of 11+ and to organise a much better transfer system, in both directions, between the modern schools and the grammar schools.

Transfer from the grammar schools probably offered no great difficulties. Transfer to the grammar school was rendered difficult, as he had already mentioned, by the foreign language and by mathematics. If, by a readjustment of the grammar school mathematics syllabus, anything could be done to ease the uphill struggle of pupils transferred from modern schools, then those who taught mathematics in grammar schools would have to face those changes: provided that they would not involve a lowering of the ultimate achievements of grammar school pupils. The speaker emphasised this last point, since the better public schools would continue to set a high standard, and the grammar schools must remain in a position to compete with them at the university scholarship level.

It seemed to the speaker that the first step was to make better contact with those who taught mathematics in modern schools. Far too few members of the Association had experience in modern schools, and it seemed to him that far too many modern school teachers had never seriously heard of the Mathematical Association. He suggested a vigorous recruiting campaign, at least for associate members, in the modern schools.

Mr. A. W. Riley (Wolverhampton) said he happened to have had the privilege for the past six years of working as an inspector in a medium-sized county borough. As such he had had to deal with many thousands of the children who formed the population of the modern schools.

Mr. Riley wished to re-direct the attention of the meeting to what he believed Mr. Swan intended to be his text: "the child in the midst." Which child? Several speakers had implied that the child who was to be in the midst was the child who was the normal inhabitant of the grammar school.

The question of an agreed syllabus in order to effect transfers at 13 had received a good deal of attention but he felt it was really a side-issue; it had nothing to do directly with the job of teaching mathematics in modern schools, whatever system of transfer might be devised between school and school, or between form and form inside one school. Those concerned would always be faced with difficulties: border-line cases which had to receive special treatment. The new Education Act would fail in its effect—and this applied not only to mathematics, but to any subject—if the work of the modern school were governed by the possible transfer of a relatively small number of children at the age of 13. The White Paper referred to the emancipation of the ordinary child; it referred to the burden under which the ordinary child (who was not going to get into the grammar school) worked. That child constituted 85 per cent. of the population of the country (whose votes, in the long run, settled everything!). The ordinary child was going to be freed, in the first place, from the burden of the examination at 11—an examination designed to select a small percentage of children; it was hoped this freedom would be realised, but a number of things would have to be attended to first.

Mr. Smith had wondered about intelligence tests, which would form a

subject for a complete discussion, though the speaker could assure members that in certain places the machinery of intelligence tests received as careful and as highly mathematical attention as the work at Rothamsted Experimental Station on which Mr. Finney had spoken during the morning. There was a much more important obstacle to be got over, and that was the attitude of the parents. If it were possible to give the parents a choice of school, as some education authorities had attempted to do, by circularising parents of the children in an age-group to find out the type of school they would like their children to attend, it would probably be found that 90 per cent. would wish their children to go to a grammar school. The reasons for that were fairly obvious ; and it was also obvious that the answer did not get one very far, because nobody would pretend that a grammar school education was suitable for 90 per cent. of the population. The idea of giving the child " the best chance " had in it the fallacy that the best chance—for *all* children—could only be had in the grammar school.

The Education Act laid down that each child must be educated according to his age, ability and aptitude, which brought one back to the frequency distribution curve. The difficulty the speaker had experienced in dealing with intelligence tests was that they were designed for a specific purpose—and in spite of any argument to the contrary there was substantial evidence that they did serve their purpose, namely to determine the selection of children who would profit by an *academic* type of education. There was, however, a danger in thinking that the tests measured the all-over intelligence, including some non-verbal factors, of the individual or even of the number of individuals in a group. If one followed Alexander's analysis of intelligence it would be found that he postulated a " g " factor, which loomed very large in mathematics. As Dr. Bickley had remarked when opening the discussion on " Geometrical Drawing in the H.S.C. Course in Schools ", the thought element in mathematics was considerable. Also there was a large proportion of the " v ", or verbal, element, and the usual group tests of intelligence had a verbal loading. Investigations of tests of the practical factor of intelligence had already been initiated ; they would take many years to complete, but the results would have considerable importance to teachers of mathematics, as of other subjects.

The speaker added that he was reminded of a book entitled *The Education of the Ordinary Child* ; on opening it, one might feel somewhat horrified to discover that the author was headmaster of a school for mental defectives. The point was that the education carried on in such schools was not trammelled by any outside considerations. The school for mental defectives was probably the only type of school where the child was really " in the midst " —where the child was given what was good for him without *first* thought being given to parents, School Certificate or future career.

As regards staff, Mrs. Williams had already said a good deal of what the speaker himself would have said, but as to the position of the Mathematical Association, there was one thing that could be done which had not yet been mentioned. When young teachers commenced their work in school it often had to be pointed out to them that they belonged to the 15 per cent. who *did* go to grammar schools ; they were not members of the " common herd " —the speaker hoped he would not be misunderstood in saying that. During their passage through grammar schools it could be pointed out to those who might become teachers in non-academic schools that they would find it necessary to modify their thoughts about the type of mathematics which would be useful to them as professional equipment ; to that end some steps might be taken in the VIth form syllabus for non-specialists. During the discussion on " Geometrical Drawing " the speaker had been glad to hear a

member from Edge Hill Training College remark that her college had developed a certain type of practical mathematics. Other colleges were experimenting with similar developments, but there was still a tremendous amount to be done in that direction. There could be no harm in starting similar experiments in the grammar schools whence came most potential teachers.

Mr. G. L. Parsons (Merchant Taylors') pointed out that at the last meeting of the Council a sub-committee consisting of Mr. Snell, Professor Broadbent and the speaker and certain other members had been appointed to go into the question of recruiting for the Association. The sub-committee had certain proposals to put forward, and if any member would like to write to him on the matter and make suggestions, particularly as to the methods of getting into touch with teachers in secondary (modern) schools, he and the other members of the sub-committee would only be too delighted.

Mr. H. V. Lowry (Woolwich Polytechnic) : In the remarks that follow I am concerned mainly with children of ages 11 to 13.

The discussion this afternoon has been far too much concerned with the child on the border-lines between the different types of secondary school. Surely it is far more important to consider first the vast majority of secondary modern school (S.M.S.) children who will not be suitable at any age for transfer to a secondary technical or grammar school. It is not very serious for a child to lose one year at the ages of 12 or 13, but it is serious if an unsuitable syllabus is thrust on the majority.

From my experience of testing boys for entry to an A.T.C. squadron, I feel sure that a very large proportion of the S.M.S. children will not know how to *use* elementary arithmetic, and also that they will be able to learn to use it if given an incentive equivalent to that supplied by the R.A.F. to boys in the A.T.C. We must try and develop such incentives as showing children the uses of arithmetic in business, factory, the home, etc., by visits to works and firms of all kinds, and thereby connecting up the use of arithmetic with the senses of touch, sight and hearing. Thus I feel that the majority of S.M.S. children should spend the ages of 11 to 13 relearning arithmetic, *i.e.* the first half of the Yorkshire syllabus together with statistical graphs, tables and numerical trigonometry.

I believe it would be a crime to try and make these children do symbolic work at all in the years 11 to 13 (and for most of them throughout the whole of their time in the S.M.S.). Possibly the situation may change a little when there are far smaller classes in primary schools, but is it really desirable that all children should do algebra even if they can understand it a little? We should surely be glad that there is as great a variety as possible amongst children, from those who do no algebra at all to mathematical specialists ; we must avoid putting the child mind in uniform. If I am right, a common syllabus from 11 to 13 for all secondary schools is quite impossible.

In disagreeing with the Yorkshire proposal of a common syllabus, I do not wish it to be inferred that the Yorkshire syllabus is badly put out. If there is to be a common syllabus the Yorkshire one is probably as good as we shall get, and although, of course, every mathematics teacher should try to weave arithmetic, algebra and trigonometry into one subject, the only clear way to state a syllabus is to divide mathematics up in the old-fashioned way.

Mr. H. H. Watts said that he must first give an extract from the preamble to the Report of the Yorkshire Branch, which he ought to have read when detailing the syllabus : " No suggestions are made as to the order in which the topics should be studied. Variety in this matter should, and does, express the individuality of the teacher, but the Committee would stress the necessity of coordinating the different branches of the subject and of emphasising their

inter-relations." He admitted that the syllabus for the first two years looked academic, but it was only the skeleton and could be filled in in many ways. Mrs. Williams would forgive him if he said that her ideas sounded attractive, but were not easy to apply. For one thing, they demanded very able teaching; were the teachers available? Again, the teaching of the subject as a whole presented difficulties. Coordination was possible, but the differences in the levels of the same topic in arithmetic, algebra and geometry were not to be despised. Finally, the grammar school pupil must be trained for further mathematics, and not sacrificed altogether for the benefit of others; these pupils deserved consideration too.

The Yorkshire Branch submitted their syllabus as a reasonable "common core", the mathematics which should be dealt with in all schools. But it was only a statement of topics and could be developed in various ways; it was a framework, dress it how you would.

Mr. C. T. Lear Caton expressed his pleasure on hearing Mr. Inman bring up for consideration what had been, the speaker felt sure, in the minds of all who had contributed to the discussion—the fact that the common curriculum might involve disadvantage to the present grammar school pupils. That did not necessarily follow. There would be in the modern schools from the age of 11, pupils having a school life of four years; in the grammar school the pupils would have a school life of at least five years from the age of 11. In those first two years a common curriculum would be suitable, and not a disadvantage, to any class of pupil; after those two years were over the grammar school pupil would have three further years, and that, surely, was the time when he could do more specialist training in mathematics. The first two years could be spent in more practical work and in more of a general education.

Mr. Smith had put in a plea that mathematics be taught as a unified subject, with which there would be general agreement. But then Mr. Smith went on to mention an illustration: that of teaching equal chords equidistant from the centre by means of a little bit of calculation. The speaker felt that illustrated a principle and at the same time had a bearing on the phrase "practical in approach", which perhaps needed defining. The theorem that equal chords were equidistant from the centre of a circle was self-evident to anyone with any degree of mathematical intuition. But it was also interesting to note that it could be derived simply from the definition of a circle and the facts of congruent triangles. If it could not be so seen, then it was not worth while trying it by means of calculation. It was better to use calculation on something which had a more useful bearing.

Mr. F. J. Swan thought there had been far too much pre-occupation with the smallest part of the school population. Surely the members were not concerned with what amounted to a twentieth or even a fiftieth; they were vitally concerned with the bulk of the pupils. Transfer was incidental. He had mentioned it because it was part of the problem.

With regard to Mrs. Williams' comments on number and space, he had put number first because, in spite of what Mrs. Williams had said, he felt that one had to start with number consciousness, although space might follow very quickly. If there was an attempt to do the two together at the beginning there might be failure. But that was a matter he and Mrs. Williams could talk out in some other place. The speaker supported what Mrs. Williams had said as to mathematics teachers with other experience. There had been in the emergency training colleges an excellent opportunity of gauging the value of the teachers who would come to the schools with experience of the outside world. Mr. Swan said he knew there had been much discussion of how the half-baked would be received by the hard-boiled, but he

could assure members that some of the teachers who would be coming to them would be very thoroughly baked ; they would be men whose experience would count.

Mr. Swan added that he was very doubtful about the " common core ". He had asked a number of students how much of the mathematics which they learned at secondary grammar schools had been useful to them in their careers. Many had followed careers which were supposed to demand a knowledge of mathematics. He had been shocked, and he thought others would be also, to learn what a small advantage those students had derived from the mathematics they had learnt in secondary grammar schools.

Finally, there still seemed too great a tendency to base conclusions on insufficient evidence. As he had said in his opening remarks, it would be well to drop the idea that opinions, not based on experiment or experience, were of any value at all ; it was only by long experiment that it would be possible to get down to the job and find an integrated syllabus for a whole child. He thanked Mr. Riley for calling the meeting back to order, in that the central theme of all such discussions must be " the child in our midst ".

The **Chairman (Mr. A. W. Siddons)** said he came down on the side of Mrs. Williams in the matter of space and number. He held that a child started to learn geometry a few months after birth—babies had a habit of feeling round for things. A child, aged 2, who could just count up to three, was very meticulous when playing with bricks, in that he always put them out straight. That seemed to show that geometry preceded number.

The discussion had been useful, and members were most grateful to the three opening speakers and to those who had followed them. It was hoped that all would work very hard to bring teachers in the modern schools into the Mathematical Association, and that all members of it would do all in their power to help those other teachers.

GLEANINGS FAR AND NEAR.

1504. Mapledurham Mill is said to be the oldest on the Thames. It is certainly the most picturesque . . . The church, too, fits most opportunely into " the picture " for the tower, appearing in the background, forms the apex of a triangle which has the mill for its base. This mathematical figure, however disguised, is, as Sir Charles Holmes has said " the secret of almost all stable and compact pictorial designs ".—Robert Gibbings, *Sweet Thames Run Softly*, pp. 111–112. [Per Mr. F. W. Kellaway.]

1505. The sea itself is vast, but so far as the visibility of surface ships is concerned it is only three dimensional. The air has the fourth " visibility " dimension of depth (or height if you like), the vastness of the sea multiplied a thousand fold up to the ceiling of high-powered aircraft . . .—Leo Walmsley ; *Fishermen at war* (London, 1941), p. 134. [Per Mr. F. Puryer White.]

1506. There was Major Hunter, a haunted little man of figures, a little man who, being a dependable unit, considered all other men either as dependable units or as unfit to live. Major Hunter was an engineer, and except in case of war no one would have thought of giving him command of men. For Major Hunter set his men in rows like figures and he added and subtracted and multiplied them. He was an arithmetician rather than a mathematician. None of the humour, the music, or the mysticism of higher mathematics ever entered his head. Men might vary in height or weight or colour, just as 6 is different from 8, but there was little other difference. He had been married several times and he did not know why his wives became very nervous before they left him.—John Steinbeck, *The Moon is Down*. [Per Mr. E. H. Lockwood.]

ON CONVERSE THEOREMS OF SUMMABILITY.

BY C. T. RAJAGOPAL.

1. This paper, while it does not merit the attention of workers in the field of summability, may be of some interest to readers of the *Gazette* who are using Bromwich's classical work on infinite series [**2**] * as a textbook. Its main purpose is to show that Bromwich's proof of the Hardy-Landau converse of Cauchy's limit theorem [**2**, § 151] can be made to yield, without any modification of the underlying idea (due to A. E. Jolliffe), all the familiar converse theorems on Rieszian summability of the first order. The paper also shows that after having obtained these theorems, one can pass on, by a natural transition suggested by Szász [**5**, Theorem 1′], to the corresponding results in the theory of summability by Dirichlet's series.

1.1. NOTATION. Let $\sum_{\nu=1}^{\infty} a_\nu$ be a (real) series and $s_n = \sum_{\nu=1}^{n} a_\nu$, $n = 1, 2, 3, \ldots$, its partial sums. Let $\{\lambda_n\}$ be the sequence :

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots, \quad \lambda_n \to \infty \ (n \to \infty).$$

Then, following Karamata [**5**], we shall say the series $\sum_{\nu=1}^{\infty} a_\nu$ or the sequence $\{s_n\}$, is summable-$R(\lambda_n, 1)$, to the sum s, if

$$\sigma_n = \frac{\sum_{\nu=1}^{n} (\lambda_{\nu+1} - \lambda_\nu) s_\nu}{\lambda_{n+1}} \to s ;$$

and summable-$D(\lambda_n)$ to the same sum, if

$$F(t) = \sum_{\nu=1}^{\infty} a_\nu e^{-\lambda_\nu t} \text{ is convergent for } t > 0,$$

$$\lim_{t \to +0} F(t) = s.$$

2. The following theorem which seems to have been first noticed by Fekete and Winn [**3**, footnote † to p. 494] furnishes a nucleus round which one may gather most of the converse theorems on $R(\lambda_n, 1)$-summability.

THEOREM A. *If the series Σa_n is summable-$R(\lambda_n, 1)$ to the sum s, the condition*

$$a_n \geqslant -H(\lambda_n - \lambda_{n-1})/\lambda_n \quad (0 < H < \infty) \qquad \text{(a)}$$

ensures

$$\lim_{n \to \infty} s_n = s.$$

The only result we require in the proof of this theorem is the simple

LEMMA 1. $$\sum_{n=m+1}^{m+p} \frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1}} < \log \frac{\lambda_{m+p+1}}{\lambda_{m+1}} < \sum_{n=m+1}^{m+p} \frac{\lambda_{n+1} - \lambda_n}{\lambda_n},$$

obtained by summing, from $n = m + 1$ to $n = m + p$, the inequalities

$$\frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1}} < \int_{\lambda_n}^{\lambda_{n+1}} \frac{dx}{x} < \frac{\lambda_{n+1} - \lambda_n}{\lambda_n}.$$

* Arabic numerals in heavy type (within square brackets) indicate the references given at the end.

2.1. Proof of Theorem A. Let us write

$$\lim_{n\to\infty} s_n = \underline{s}, \quad \overline{\lim_{n\to\infty}}\, s_n = \bar{s},$$

and suppose (as we may, without loss of generality) $\underline{s} = 0$. Then

$$\underline{s} \leqslant 0 \leqslant \bar{s},$$

and consequently it is sufficient to prove that $\bar{s} > 0$ results in a contradiction.

First, we can find n_0 so that for $n > n_0$, $|\sigma_n| < \epsilon$ (arbitrarily small).

If $\underline{s} > 0$, we can find $m > n_0$, $K > 0$, so that

$$s_{m+1} > 2K, \quad \min(s_{m+2}, s_{m+3}, \dots, s_{m+p}) \geqslant K, \ s_{m+p+1} < K,$$

where $p \geqslant 1$. (If $p = 1$, then $s_{m+1} > 2K$, $s_{m+2} < K$.) Hence

$$s_{n+1} - \sigma_{n+1} > K - \epsilon \quad (n = m, m+1, \dots, m+p-1).$$

Now, the identity

$$\sigma_{n+1} - \sigma_n = \frac{\lambda_{n+2} - \lambda_{n+1}}{\lambda_{n+1}}(s_{n+1} - \sigma_{n+1}) \qquad \dots\dots(1)$$

gives on summation

$$2\epsilon > \sigma_{m+p} - \sigma_m = \sum_{n=m}^{m+p-1} \frac{\lambda_{n+2} - \lambda_{n+1}}{\lambda_{n+1}} (s_{n+1} - \sigma_{n+1})$$

$$> (K - \epsilon) \sum_{n=m}^{m+p-1} \frac{\lambda_{n+2} - \lambda_{n+1}}{\lambda_{n+1}}$$

$$> (K - \epsilon) \log \frac{\lambda_{m+p+1}}{\lambda_{m+1}}, \qquad \dots\dots(2a)$$

by Lemma 1.

Again, from our choice of m, p and (a),

$$K < s_{m+1} - s_{m+p+1} \leqslant H \sum_{n=m+1}^{m+p} \frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1}}, \qquad \dots\dots(3a)$$

$$< H \log \frac{\lambda_{m+p+1}}{\lambda_{m+1}}, \qquad \dots\dots(4a)$$

by Lemma 1.

(2a) and (4a) make $2H\epsilon > (K - \epsilon)K$ or $\epsilon > K^2/(2H + K)$, contradicting the fact that ϵ is arbitrarily small, and thus completing the proof.

2.2. If (a) is true as it is, and also with a_n changed to $-a_n$, § 2.1 gives at once $\bar{s} = 0 = \underline{s}$, and therefore

Theorem B. *If the series Σa_n is summable-$R(\lambda_n, 1)$, the hypothesis*

$$|a_n| \leqslant H(\lambda_n - \lambda_{n-1})/\lambda_n \qquad \dots\dots(b)$$

ensures the convergence of the series.

3. This section gives all the familiar alternatives to (b), any one of which can be substituted for (b) in Theorem B without affecting the theorem otherwise.

3.1. If, in § 2.1, $\underline{s} < 0$, we can find $m' > n_0$, $K' > 0$, so that

$$s_{m'+1} > -K', \ \max(s_{m'+2}, s_{m'+3}, \dots, s_{m'+p'}) \leqslant -K', \ s_{m'+p'+1} < -2K'.$$

It follows, as before,

$$2\epsilon > \sigma_{m'+1} - \sigma_{m'+p'} > (K' - \epsilon) \log(\lambda_{m'+p'+1}/\lambda_{m'+2}). \qquad \dots\dots(2c)$$

Provided $\lim(\lambda_{n+1}/\lambda_n) = 1$, we have also

$$K' < s_{m'+1} - s_{m'+p'+1} < H\left(\log\frac{\lambda_{m'+p'+1}}{\lambda_{m'+2}} + \log\frac{\lambda_{m'+2}}{\lambda_{m'+1}}\right) \quad \text{.....(3c)}$$

$$< H\left(\log\frac{\lambda_{m'+p'+1}}{\lambda_{m'+2}} + \frac{\epsilon}{K'-\epsilon}\right) \quad \text{.....(4c)}$$

if m' is sufficiently large. Since (2c) and (4c) make $\epsilon > K'^2/(3H + K')$, it follows that $s = 0$. We have thus

THEOREM C. *Theorem B remains true when* (b) *is changed to*

$$a_n \geqslant -H(\lambda_n - \lambda_{n-1})/\lambda_n, \quad \lim_{n\to\infty}(\lambda_{n+1}/\lambda_n) = 1. \quad \text{.....(c)}$$

3.2. Next, if $\underline{s} < 0$ and $\underline{\lim}\,(s_n - s_{n-1}) \geqslant 0$, we have, in addition to (2c), (3c) in the modified form :

$$K' < s_{m'+1} - s_{m'+p'+1} < H\log\frac{\lambda_{m'+p'+1}}{\lambda_{m'+2}} + s_{m'+1} - s_{m'+2} \quad \text{.....(3d)}$$

$$< H\left(\log\frac{\lambda_{m'+p'+1}}{\lambda_{m'+2}} + \frac{\epsilon}{K'-\epsilon}\right).$$

Hence, arguing as in § 3.1, we are led to

THEOREM D. *Theorem B is true with* (b) *replaced by*

$$a_n \geqslant -H(\lambda_n - \lambda_{n-1})/\lambda_n, \quad \underline{\lim_{n\to\infty}}\, a_n \geqslant 0. \quad \text{.....(d)}$$

It may be pointed out that, in §§ 3.1, 3.2, we have tacitly assumed $p' > 1$. If $p' = 1$, we have a trivial case in which we can discard (2c) and draw our conclusion from either (3c) or (3d) alone.

3.3. Again, with the supposition $s < 0$, as in § 3.1,

$$2\epsilon > \sigma_{m'+1} - \sigma_{m'+p'+1} > (K' - \epsilon)\log(\lambda_{m'+p'+2}/\lambda_{m'+2}). \quad \text{.....(2e)}$$

Also, we have, if we assume $a_{n-1} \geqslant -H(\lambda_n - \lambda_{n-1})/\lambda_n$,

$$K' < s_{m'+1} - s_{m'+p'+1} \leqslant H\sum_{n=m'+2}^{m'+p'+1}\frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1}} \quad \text{.....(3e)}$$

$$< H\log\frac{\lambda_{m'+p'+2}}{\lambda_{m'+2}}. \quad \text{.....(4e)}$$

It is now seen, as in § 2.1, that (2e) and (4e) contradict the fact that ϵ is arbitrary, and therefore give rise to

THEOREM E. *Theorem B is true with the following condition in place of* (b) :

$$a_n \geqslant -H(\lambda_n - \lambda_{n-1})/\lambda_n, \quad a_{n-1} \geqslant -H(\lambda_n - \lambda_{n-1})/\lambda_n. \quad \text{.....(e)}$$

For a detailed account of convergence conditions for series summable-$R(\lambda_n, 1)$ and a complete bibliography, the reader is referred to **Karamata**'s paper [**6**] and pamphlet [**5**], cited at the end.

4. The list of propositions given above can be made fairly complete if we add to it the next two theorems, which are direct consequences of the definition of $R(\lambda_n, 1)$-summability.

4.1. THEOREM F. *Suppose* $\underline{\lim}\,(\lambda_{n+1}/\lambda_n) > 1$. *Then*

(i) $\underline{\lim}\,(\sigma_{n+1} - \sigma_n) = 0$ *implies* $\underline{\lim}\, s_n = \underline{\lim}\, \sigma_n$;

(ii) $\lim\,(\sigma_{n+1} - \sigma_n) = 0$ *implies* $\operatorname{Osc} s_n = \operatorname{Osc} \sigma_n$;

(iii) $R(\lambda_n, 1)$*-summability of* $\{s_n\}$ *implies the convergence of* $\{s_n\}$.

PROOF. (i) follows from the fact that, on account of the relation (1) of § 2.1, $\underline{\lim}\,(\sigma_{n+1} - \sigma_n) = 0$ and $\underline{\lim}\, s_n < \underline{\lim}\, \sigma_n$ together involve $\underline{\lim}\,(\lambda_{n+1}/\lambda_n) = 1$.

(ii) is the result of applying (i) to $\{s_n\}$ and $\{-s_n\}$.

(iii) is the special case of (ii) in which $\operatorname{Osc} \sigma_n = 0$.

Theorem F shows that Theorems A, B, D, E are non-trivial only when $\lim\,(\lambda_{n+1}/\lambda_n) = 1$.

4.2. THEOREM G. *Necessary and sufficient conditions for the convergence of* Σa_n *are the* $R(\lambda_n, 1)$*-summability of the series and the condition*

$$\sum_{\nu=1}^{n} a_\nu \lambda_\nu = o(\lambda_n), \quad n \to \infty . \qquad \text{.....(g)}$$

PROOF. That (g) is necessary for the convergence of $\Sigma\, a_n$ follows from the use of the Cauchy-Stolz limit theorem [2, § 147] in the relation

$$\frac{\sum_{\nu=1}^{n} a_\nu \lambda_\nu}{\lambda_n} = s_n - \frac{\sum_{\nu=1}^{n-1} s_\nu(\lambda_{\nu+1} - \lambda_\nu)}{\lambda_n} .$$

That the $R(\lambda_n, 1)$-summability of $\Sigma\, a_n$ or $\{s_n\}$ is necessary follows similarly from the expression for σ_n.

Next, the expression for σ_n written in the form

$$\sigma_n = s_n - \frac{\sum_{\nu=1}^{n} a_\nu \lambda_\nu}{\lambda_{n+1}} ,$$

shows that the conditions given in the theorem are sufficient for the convergence of $\Sigma\, a_n$.

5. The foregoing theorems, with the exception of Theorem F, when taken in conjunction with the result of Szász already referred to, yield straight away the corresponding theorems for $D(\lambda_n)$-summability. Szász's result is quoted here as

LEMMA 2. $D(\lambda_n)$*-summability of* $\Sigma\, a_n$ *ensures* $R(\lambda_n, 1)$*-summability of* $\Sigma\, a_n$, *provided*

$$\sum_{\nu=1}^{n} a_\nu \lambda_\nu \geqslant -H\lambda_n .$$

Since the last condition is implied by condition (a) and *a fortiori* by any one of the conditions (b)–(e), we have at once :

THEOREMS A′–E′. *Theorems* A–E *can be restated with the hypothesis of* $D(\lambda_n)$*-summability of* $\Sigma\, a_n$ *substituted for that of* $R(\lambda_n, 1)$*-summability of* $\Sigma\, a_n$ *and without any other change.*

In much the same way we can deduce from Theorem G, Schnee's

THEOREM G′. *Necessary and sufficient conditions for the convergence of* $\Sigma\, a_n$ *are the* $D(\lambda_n)$*-summability of the series and the condition* (g).

PROOF. The necessity part of the theorem can be proved directly.

To prove the sufficiency part we observe that (g) ensures

$$\left| \sum_{\nu=1}^{n} a_\nu \lambda_\nu \right| \leqslant H\lambda_n ,$$

which, together with the $D(\lambda_n)$-summability of Σa_n, implies the $R(\lambda_n, 1)$-summability of Σa_n. Then we have merely to appeal to Theorem G to complete our proof.

6. The $D(\lambda_n)$-analogue of Theorem F (iii), excluded from consideration in § 5, is

THEOREM F′. *If* $\underline{\lim}\,(\lambda_{n+1}/\lambda_n) > 1$, *$D(\lambda_n)$-summability of Σa_n ensures the convergence of Σa_n.*

The main step in the deduction of Theorem F′ from Theorem F (iii), using Lemma 2—or, for that matter, in any direct proof of Theorem F′—is the following lemma of which a simple proof has been given by Ingham [**4**, II].

LEMMA 3. *If* $\underline{\lim}\,(\lambda_{n+1}/\lambda_n) > 1$, *then* $|F(t)| \leqslant M$ *for* $t > 0$ *implies* $a_n = O(1)$.

The implication $a_n = O(1)$, taken along with the fact that there is a constant $c > 1$ such that $\lambda_{n+1}/\lambda_n \geqslant c\,(n \geqslant 1)$, gives at once

$$\frac{\sum_{\nu=1}^{n} a_\nu \lambda_\nu}{\lambda_n} = O\left(\frac{1}{c^{n-1}} + \frac{1}{c^{n-2}} + \ldots + 1\right) = O(1), \quad n \to \infty .$$

Lemma 2 now shows that the hypothesis of $D(\lambda_n)$-summability of Σa_n in Theorem F′ can be replaced by that of $R(\lambda_n, 1)$-summability of Σa_n, and so this theorem can be reduced to Theorem F (iii).

Using Ingham's idea, L. S. Bosanquet has recently obtained a theorem [**1**, Theorem D] which includes both Theorems B′ and F′.

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C. T. RAJAGOPAL.

1507. CONVERSATION PIECE.

Allenby : I hear you are fining the villages in your area somewhat heavily, General X.

General X : Well, sir, when a village misbehaves itself, I fine it ten per cent. of its ghaffir tax.

Allenby : That's not what I've heard, X. I'm told you fine them ten times their ghaffir tax.

General X : Yes, that's right, sir—ten per cent.

Allenby : But that's not ten per cent. ; that's a thousand per cent.

General X : Oh, is it, sir ? (*Pause.*) Well, anyway, it's what I call ten per cent., and when I say ten per cent. they know what they've got to pay, and they pay it all right, sir.—Lord Wavell, *Allenby in Egypt*, p. 46. [Per Mr. E. H. Lockwood.]

A TREATMENT OF THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS.

By P. Gant.

1. *Introduction.*

Before the end of the first year of the H.S.C. course, a boy has dealt with the elementary theory of indices and logarithms and has been introduced to the exponential and logarithmic functions in Calculus. This introduction, the writer has suggested in a previous Note (1896), may well be intuitive and may be based upon the fact that if $y=a^x$, then $dy/dx=ky$, the fact being established by reading off gradients and ordinates from a suitable graph. It is now suggested that when the H.S.C. course in Calculus is completed, a mathematically-minded boy may learn a great deal of mathematical technique from a treatment of the exponential and logarithmic functions of the kind outlined below.

2. *Mode of treatment.*

New functions $L(x)$ and $E(x)$ are defined and their main properties are established from their differential equations. Maclaurin expansions are obtained for purposes of computation, but the knowledge of infinite series pre-supposed is simply that if $R_n \to 0$, then $S_n \to f(x)$ in such an expansion; there is no other appeal to the algebra of infinite series. Finally, the functions are identified with the previously-met functions $\log_e x$ and e^x.

3. *The function $L(x)$.*

This is defined for $x>0$ by the equations

$$L'(x)=1/x, \qquad L(1)=0.$$

The first equation determines $L(x)$ as an indefinite integral and the second fixes the constant of integration, so that the function is completely defined.

3.1. Since $L'(x)$ exists for all $x>0$, $L(x)$ is continuous and hence finite.

3.2. The solution of $L(x)=0$ is unique.

For, if $L(\alpha)=L(1)=0$, where $\alpha \neq 1$ then, by Rolle's Theorem, $L'(x)$ vanishes for some intermediate value of x; but $L'(x)=1/x \neq 0$, and hence a contradiction. Thus $L(x)=0$ implies $x=1$.

3.2.1. Also, since $L'(x)>0$, it follows from the Mean Value Theorem that $L(x)$ steadily increases with x.

3.2.2. Since $L(1)=0$, $L(x)>0$ for $x>1$, and $L(x)<0$ for $0<x<1$.

3.3. *Multiplication property.*

For any α, $\beta>0$,

$$L(\alpha)+L(\beta)=L(\alpha\beta).$$

Consider $$f(x)=L(\alpha x)+L(\beta x)-L(\alpha\beta x^2).$$

Here $$f'(x)=\alpha/\alpha x \ +\beta/\beta x - 2\alpha\beta x/\alpha\beta x^2 = 0.$$

On integrating, $f(x)=C$, a constant.

Put $x=1/\beta$;

$$L(\alpha/\beta)+L(1)-L(\alpha/\beta)=C.$$

Hence $C=0$.

Thus for all x,

$$L(\alpha x)+L(\beta x)-L(\alpha\beta x^2)=0.$$

On putting $x=1$, we obtain the required result.

3.3.1. On replacing α by α/β in **3.3**, we obtain

$$L(\alpha/\beta)+L(\beta)=L(\alpha),$$

or

$$L(\alpha)-L(\beta)=L(\alpha/\beta).$$

3.3.2. We may obtain the result **3.3.1** directly by considering

$$g(x)=L(\alpha x)-L(\beta x),$$

and proceeding as in **3.3.**

3.3.3. Putting $\alpha=1$ in **3.3.1** we obtain

$$L(1)-L(\beta)=L(1/\beta),$$

or

$$L(1/\beta)=-L(\beta).$$

3.4. *Integral form of definition.*

The definition of $L(x)$ is equivalent to

$$L(x)=\int_1^x dt/t, \quad x>0.$$

Hence we may obtain alternative proofs of the results of **3.3.**

3.4.1.

$$L(\alpha\beta)=\int_1^{\alpha\beta} dt/t=\int_1^{\alpha} dt/t+\int_{\alpha}^{\alpha\beta} dt/t$$

$$=L(\alpha)+\int_1^{\beta} du/u, \text{ where } t=\alpha u$$

$$=L(\alpha)+L(\beta).$$

3.4.2.

$$L(\alpha/\beta)=\int_1^{\alpha/\beta} dt/t=\int_1^{\alpha} dt/t+\int_{\alpha}^{\alpha/\beta} dt/t$$

$$=L(\alpha)-\int_1^{\beta} du/u, \text{ where } t=\alpha/u$$

$$=L(\alpha)-L(\beta).$$

3.5. If $f(x)>0$, then

$$\frac{d}{dx}[L\{f(x)\}]=\frac{f'(x)}{f(x)}.$$

Hence

$$\int\frac{f'(x)dx}{f(x)}=L\{f(x)\}.$$

3.6. *Maclaurin expansions.*

$$L'(1+x)=1/(1+x),$$

$$L^{(n)}(1+x)=(-)^{n-1}(n-1)!/(1+x)^n.$$

Hence, using the Cauchy form of the remainder, we obtain

$$L(x)=x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+\ldots. \quad (-1<x\leqslant 1).$$

3.6.1. Also, $L\{(1+x)/(1-x)\}=L(1+x)-L(1-x)$, by **3.3.1**,

and so

$$L^{(n)}\{(1+x)/(1-x)\}=L^{(n)}(1+x)-L^{(n)}(1-x).$$

Hence as above we obtain

$$L\{(1+x)/(1-x)\}=2\{x+\tfrac{1}{3}x^3+\tfrac{1}{5}x^5+\ldots\}, \quad (-1<x<1).$$

Note that we do not subtract one infinite series from another to obtain the latter expansion.

3.6.2. Hence we obtain expansions for $L(m/n)$ and $L\{(p+1)/p\}$, from which the function may be computed.

3.7. *Limits as $x \to 0$ and $x \to \infty$.*

In 3.1 and 3.2 it was established that $L(x)$ is continuous, finite, negative for $0 < x < 1$, positive for $x > 1$, and steadily increasing with x. The limits as $x \to +\infty$ and as $x \to 0$ were not investigated.

3.7.1. By 3.3, $L(\alpha^n) = n \,.\, L(\alpha)$ for $\alpha > 0$ and n a positive integer.

Hence if $\alpha > 1$, so that $L(\alpha) > 0$, then $L(\alpha^n) \to +\infty$ as n, and so $\alpha^n \to +\infty$.
Thus $L(x) \to +\infty$ as $x \to +\infty$.

3.7.2. Similarly, if $0 < \alpha < 1$, so that $L(\alpha) < 0$, then $L(\alpha^n) \to -\infty$ as $n \to +\infty$ and $\alpha^n \to 0$.

Hence $L(x) \to -\infty$ as $x \to 0$.

3.7.3. Thus $L(x)$ steadily increases from $-\infty$ to $+\infty$ as x increases from 0 to ∞.

3.8. *Inverse of $L(x)$.*

From 3.7.3, we see that $L(x)$ has an inverse function, $E(x)$, which increases steadily from 0 to $+\infty$ as x increases from $-\infty$ to $+\infty$. It is formally expressed by the equation

$$L\{E(x)\} = x.$$

3.8.1. On differentiating the equation 3.8 we obtain

$$E'(x)/E(x) = 1, \qquad \text{by 3.5.}$$

Hence $$E'(x) = E(x), \text{ for all } x.$$

3.8.2. Put $x = 0$ in the equation 3.8 ;

$$L\{E(0)\} = 0.$$

Hence, $$E(0) = 1, \quad \text{by 3.2.}$$

4. *The function $E(x)$.*

This is defined by the equations

$$E'(x) = E(x), \qquad E(0) = 1,$$

and so is identical with the inverse of $L(x)$ discussed in 3.8.

4.1. *The addition theorem.*

For any α, β,

$$E(\alpha) \,.\, E(\beta) = E(\alpha + \beta).$$

Consider $$f(x) = E(\alpha x) \,.\, E(\beta x),$$

and $$g(x) = E\{(\alpha + \beta)x\}.$$

Then
$$\begin{aligned} f'(x) &= \alpha \,.\, E'(\alpha x) \,.\, E(\beta x) + E(\alpha x) \,.\, \beta \,.\, E'(\beta x) \\ &= (\alpha + \beta) f(x), \end{aligned}$$

and
$$\begin{aligned} g'(x) &= (\alpha + \beta) E'\{(\alpha + \beta)x\} \\ &= (\alpha + \beta) g(x). \end{aligned}$$

Thus $$f'(x)/f(x) = \alpha + \beta = g'(x)/g(x).$$

Now
$$\begin{aligned} \frac{d}{dx}[f(x)/g(x)] &= (gf' - fg')/g^2 \\ &= \frac{f}{g}\left[\frac{f'}{f} - \frac{g'}{g}\right] \\ &= 0, \end{aligned}$$

since $g \neq 0$, by 3.8.

On integrating, $f(x)/g(x) = C$,
and putting $x = 0$,

$$E(0) \,.\, E(0)/E(0) = 1 = C.$$

Hence $f(x) = g(x)$, for all x.

Thus $E(\alpha x) \,.\, E(\beta x) = E\{(\alpha + \beta)x\}$, for all x.

On putting $x = 1$, the result follows.

4.1.1. Alternatively, we may proceed as follows, from the stage

$$f'(x)/f(x) = g'(x)/g(x).$$

$$\int f'(x)/f(x) \,.\, dx = \int g'(x)/g(x) \,.\, dx.$$

Since $f(x)$, $g(x) > 0$ by 3.8, we obtain, by 3.5,

$$L\{f(x)\} = L\{g(x)\} + C.$$

Put $x = 0$: by the definitions

$$L\{f(0)\} = L\{E(0) \,.\, E(0)\} = L(1) = 0,$$

$$L\{g(0)\} = L\{E(0)\} = L(1) = 0.$$

Hence $C = 0$ and

$$L\{f(x)\} - L\{g(x)\} = 0, \quad \text{for all } x.$$

By 3.3.1, $L\{f(x)/g(x)\} = 0$, for all x,

and by 3.2, $f(x)/g(x) = 1$, for all x,

or $f(x) = g(x)$, for all x.

Thus $E(\alpha x)E(\beta x) = E\{(\alpha + \beta)x\}$, for all x,

and on putting $x = 1$ we obtain 5.1.

4.2. *Maclaurin expansion.*

$$E(0) = E'(0) = \ldots = E^{(n)}(0) = 1.$$

Also, $E^{(n)}(\theta x) = E(\theta x) < E(|\, x \,|)$, where $0 < \theta < 1$, since $E(x)$ steadily increases with x.

Hence, using the Lagrange form of remainder,

$$\begin{aligned} |\, R_n \,| &= |\, x^n E^{(n)}(\theta x)/n! \,| \\ &< |\, x^n/n! \,| \,.\, E(|\, x \,|) \\ &\to 0 \text{ as } n \to \infty \text{ for any } x. \end{aligned}$$

Hence we obtain, for all x,

$$E(x) = 1 + x + x^2/2! + x^3/3! + \ldots,$$

from which the function may be computed.

5. *Relation of $E(x)$ to e^x.*

We define e by the equation

$$e = E(1),$$

which may be computed from 4.2.

Then $E(2) = E(1 + 1) = E(1) \,.\, E(1)$, by 4.1
$= e^2$;

and for n a positive integer,

$$E(n) = e^n.$$

5.1. $\{E(p/q)\}^q = E(p/q) \,.\, E(p/q) \ldots$ to q factors
$= E(p/q + p/q + \ldots$ to q terms)
$= E(p)$
$= e^p$.

Hence $E(p/q) = \sqrt{e^p} = e^{p/q}$, for positive integers p, q.

5.2. $1 = E(0) = E\{m + (-m)\}$
$= E(m) \,.\, E(-m)$, by 4.1,
$= e^m \,.\, E(-m)$.

Hence $E(-m) = 1/e^m = e^{-m}$, for $m > 0$.

5.3. For irrational x we define e^x as $E(x)$.

5.4. Thus $E(x)$ and e^x are identical.

6. *Relation of* $L(x)$ *to* $\log_e x$.

From the results (i) $E(x)$ and e^x are identical;
(ii) $E(x)$ and $L(x)$ are inverse;
(iii) e^x and $\log_e x$ are inverse,

we conclude that $L(x)$ and $\log_e x$ are identical. P. GANT.

1508. She liked numbers and sums. She devised a game in which each number was a family member and the "answer" made a family grouping with a story to it. Naught was a babe in arms. He gave no trouble. Whenever he appeared you just "carried" him. The figure 1 was a pretty baby girl just learning to walk, and easy to handle; 2 was a baby boy who could walk and talk a little. He went into family life (into sums, etc.) with very little trouble. And 3 was an older boy in kindergarten, who had to be watched a little. Then there was 4, a girl of Francie's age. She was almost as easy to "mind" as 2. The mother was 5, gentle and kind. In large sums, she came along and made everything easy the way a mother should. The father 6, was harder than the others but very just. But 7 was mean. He was a crochety old grandfather and not at all acountable for how he came out. The grandmother, 8, was hard too, but easier to understand than 7. Hardest of all was 9. He was company, and what a hard time fitting him into family life!

When Francie added a sum, she would fix a little story to go with the result. If the answer was 924, it meant that the little boy and girl were being minded by company while the rest of the family went out. When a number such as 1024 appeared, it meant that all the little children were playing together in the yard. The number 62 meant that papa was taking the little boy for a walk; 50 meant that mama had the baby out in the buggy for an airing, and 78 meant grandfather and grandmother sitting home by the fire of a winter's evening. Each single combination of numbers was a new set-up for the family, and no two stories were ever the same.

Francie took the game with her up into algebra. X was the boy's sweetheart, who came into the family life and complicated it. Y was the boy friend who caused trouble. So arithmetic was a warm and human thing to Francie and occupied many lonely hours of her time.—Betty Smith, *The Tree in the Yard* (Heinemann), p. 121. [Per Mr. H. Webb.]

1509. Actors moved in great houses and yet knew the people's temper; agile as apes, painted as tropical birds, they moved freely about the political jungle. In the great equation of the times, they alone were independent variables.—John Russell, *Shakespeare's Country* (Batsford, 1942), p. 24. [Per Mr. C. E. Kemp.]

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A LESSON ON π AND ON $\lim_{\theta \to 0} (\sin \theta/\theta)$.

By F. J. Turton.

1. (*a*) The foundation of calculus operations on the circular functions is

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1.$$

The traditional steps by which the inequality

$$\tan \theta > \theta > \sin \theta$$

(from which the limit follows) is deduced, are felt by many teachers to be unsatisfactory for foundations, at two points in the argument :

(i) It has to be assumed that the circumference of a circle varies as its radius. The " proof " of this (*e.g.* Todhunter and Hogg, *Trigonometry*, p. 8) assumes as an axiom that the perimeter of an inscribed n-sided regular polygon " can be made to differ by as little as we please " from the circumference of the circle in which it is inscribed. This in itself is not so obvious as is desirable in a fundamental proposition ; moreover, if put into trigonometrical form, it will be seen to be equivalent to an axiom that

$$\lim_{n \to \infty} \left(\sin \frac{\pi}{n} \Big/ \frac{\pi}{n} \right) = 1,$$

that is, a limit involving some of the values of θ in

$$\lim_{\theta \to 0} (\sin \theta/\theta) = 1.$$

(ii) It is later taken as an axiom (Todhunter and Hogg, *loc. cit.*, p. 82) that (Fig. 1) $BT >$ arc $BA > BM$. That BT is greater than BM is evident, and

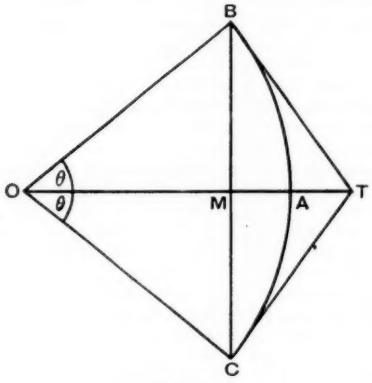

Fig. 1.

that arc BC ($= 2$. arc BA) is greater than chord BC ($= 2BM$) is equivalent to Euclid's definition of a straight line. But it is not at all obvious that

$$BT > \text{arc } BA.$$

(*b*) Teachers who are reluctant to base their calculus on this second axiom, deduce the inequality

$$\tan\theta > \theta > \sin\theta$$

from the inequality

$$\text{area triangle } OBT > \text{area sector } OAB > \text{area triangle } OAB.$$

This necessitates an expression for the area of the sector OAB. Some teachers assume the area of the whole circle to be πr^2, π having been defined by the relation " circumference $= 2\pi r$ ". This requires a further assumption : either that

$$\begin{aligned}&\lim(\text{area of polygon} : \text{perimeter of polygon})\\ &= \lim(\text{area of polygon}) : \lim(\text{perimeter of polygon})\\ &= \text{area of circle} : \text{circumference of circle},\end{aligned}$$

in which is involved, both in numerator and denominator, a limit equivalent to

$$\lim \sin\left(\frac{\pi}{n}\Big/\frac{\pi}{n}\right) = 1\ ;$$

or that

$$\begin{aligned}&\lim(\text{area of polygon} : \text{perimeter of polygon})\\ &= \text{area of circle} : \text{circumference of circle},\end{aligned}$$

which appears to be quite unwarranted.

(*c*) Alternatively, it may be assumed that π is defined by " area $= \pi r^2$ " ; this assumes that the proposition " areas of similar figures vary as the square of their linear dimensions " can be extended to circles, and seems difficult to justify except by assuming that the areas of a suitable sequence of polygons have a limit and that this limit is the area of a circle, equivalent to assuming that $\sin\frac{\pi}{n}\Big/\frac{\pi}{n}$ has a limit as $n \to \infty$.

(*d*) The treatment which follows is submitted as an attempt to put the inequality on a sounder footing. Besides attempting to overcome the objections above, it brings in the radian when required as a calculus unit for measurement of angles, defined so as to make the fundamental trigonometrical limit, $\lim_{\theta\to 0}(\sin\theta/\theta)$, as simple as possible, namely, unity ; and leaves the not very convincing axiom as to the length of the circumference to affect only the estimate of this length. Further, the definition of π here used links directly with the method used to approximate to π, by all computers from Archimedes onward, until the development of series methods.

2. (*a*) Circumscribe an equilateral triangle round a circle, ABC, of radius r (Fig. 2). It has

$$3 \text{ sides and area } 3r^2 \tan 60°.$$

Bisect the angles to find D, E, F, and draw tangents at these points to form a regular hexagon which has

$$3 \,.\, 2 \text{ sides and area } 3r^2 \,.\, 2 \,.\, \tan(60°/2).$$

Similarly form a regular polygon with 12 sides,

$$3 \,.\, 2^2 \text{ sides and area } 3r^2 \,.\, 2^2 \,.\, \tan(60°/2^2).$$

Proceeding thus, we have in the general figure

$$3 \,.\, 2^n \text{ sides and area } 3r^2 \,.\, 2^n \tan(60°/2^n).$$

It is geometrically obvious that the areas of the polygons form a sequence which decreases steadily as n increases.

(*b*) Now inscribe a regular hexagon *ADBECF* in the circle. It has

6 sides and area $3r^2 . \sin 60°$.

Bisecting its angles, construct the regular polygon with

$6 . 2$ sides and area $3r^2 . 2 . \sin (60°/2)$;

Repeating the process, we next obtain

$6 . 2^2$ sides and area $3r^2 . 2^2 \sin (60°/2^2)$,

and so on, so that we obtain

$6 . 2^n$ sides and area $3r^2 . 2^n . \sin (60°/2^n)$.

It is geometrically obvious that the areas increase steadily as n increases.

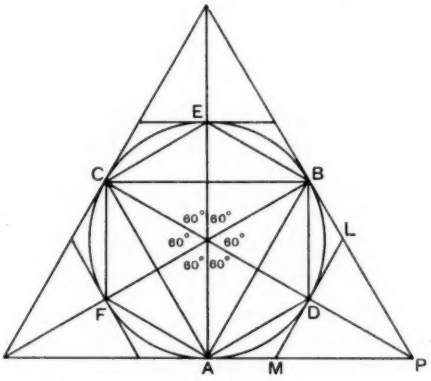

FIG. 2.

(*c*) The area of the circle is less than the area of any circumscribing polygon and greater than that of any inscribed polygon. If the area of the circle be S, then

$$3r^2 \tan 60° > 3r^2 . 2 . \tan (60°/2) > 3r^2 . 2^2 . \tan (60°/2^2) > \ldots$$
$$> 3r^2 . 2^n \tan (60°/2^n) > \ldots > S > \ldots > 3r^2 . 2^n \sin (60°/2^n) > \ldots$$
$$> 3r^2 . 2^2 . \sin (60°/2^2) > 3r^2 . 2 . \sin (60°/2) > 3r^2 \sin 60°,$$

n being an integer greater than 2.

Now $3r^2 . 2^n . \tan (60°/2^n)$ decreases as n increases, but is never less than, say, $3r^2 \sin 60°$; hence it has a limit as $n \to \infty$. Also $3r^2 . 2^n \sin (60°/2^n)$ increases as n increases, but never exceeds, say, $3r^2 \tan 60°$; hence it has a limit as $n \to \infty$.

Also

$$\lim_{n\to\infty} \{3r^2 . 2^n . \tan (60°/2^n)\} : \lim_{n\to\infty} \{3r^2 . 2^n . \sin (60°/2^n)\}$$
$$= \lim_{n\to\infty} \frac{3r^2 . 2^n \tan (60°/2^n)}{3r^2 . 2^n \sin (60°/2^n)}$$
$$= \lim_{n\to\infty} \frac{1}{\cos (60°/2^n)} = 1.$$

But S differs from $3r^2 \cdot 2^n \cdot \tan(60°/2^n)$ and from $3r^2 \cdot 2^n \sin(60°/2^n)$ by less than they differ from each other; hence

$$S = \lim_{n\to\infty} 3r^2 \cdot 2^n \cdot \tan(60°/2^n) = \lim_{n\to\infty} 3r^2 \cdot 2^n \cdot \sin(60°/2^n),$$

say $S = \pi r^2$, where $\pi = \lim_{n\to\infty} 3 \cdot 2^n \cdot \sin(60°/2^n)$. ($r$ is independent of n, and so π is independent of r.)

3. The area of any circle of radius r is πr^2. The area of any sector is axiomatically taken as proportional to the angle at the centre. Let this angle be θ (Fig. 1), in some system of units not yet fixed. Then the area of the sector OAB is $kr^2\theta$, where k is some constant. Now

$$\text{area } \triangle OBT > \text{area sector } OAB > \text{area } \triangle OAB;$$

that is,

$$\tfrac{1}{2}r^2 \tan\theta > kr^2\theta > \tfrac{1}{2}r^2 \sin\theta,$$

giving

$$1/\cos\theta > 2k(\theta/\sin\theta) > 1,$$

or

$$\cos\theta < \frac{1}{2k}\frac{\sin\theta}{\theta} < 1,$$

and this holds for $0 < \theta < 90°$.

Put $\theta = -\phi$:

$$\cos(-\phi) < \frac{1}{2k}\frac{\sin(-\phi)}{-\phi} < 1,$$

or

$$\cos\phi < \frac{1}{2k}\frac{\sin\phi}{\phi} < 1,$$

which holds for $0 > \phi > -90°$.

Hence for all angles between $-90°$ and $+90°$, except $\theta = 0$,

$$\cos\theta < \frac{1}{2k}\frac{\sin\theta}{\theta} < 1.$$

But $\lim_{\theta\to 0} \cos\theta = 1$, and hence

$$\lim_{\theta\to 0} \frac{1}{2k}\frac{\sin\theta}{\theta} = 1.$$

It will evidently be most convenient to have

$$\lim_{\theta\to 0} \sin\theta/\theta = 1,$$

and this will be the case if $k = \frac{1}{2}$.

The area of the sector with angle θ at the centre becomes $\frac{1}{2}r^2\theta$, and the angle, θ_c, at the centre for the whole circle is such that $\frac{1}{2}r^2\theta_c = \pi r^2$; hence $\theta_c = 2\pi$. Thus the new unit, the radian, in which θ is to be measured, is such that the angle at the centre of a circle, namely 4 right angles or 360°, is equal to 2π radians. As yet, π is known only as $\lim 3 \cdot 2^n \sin(60°/2^n)$ and by the fact that the area of a circle of radius r is πr^2.

4. Circumference of a circle.

The perimeter of a circumscribing equilateral triangle (Fig. 2) is $6r \tan 60°$. Forming a regular hexagon as in § 2 (a), we have

$3 \cdot 2$ sides, perimeter $6r \cdot 2 \tan(60°/2)$;

and continuing as in § 2 (a),

$3 \cdot 2^2$ sides, perimeter $6r \cdot 2^2 \cdot \tan(60°/2^2)$;

and so on, to

$3 \cdot 2^n$ sides, perimeter $6r \cdot 2^n \cdot \tan(60°/2^n)$.

It is geometrically obvious that the perimeters decrease consistently as n increases: for instance, from triangle to hexagon, $LP + PM$ is replaced by LM (Fig. 2).

Again, for the inscribed equilateral triangle:

3 sides, perimeter $6r \cdot \sin 60°$:

for the regular hexagon, as in § 2 (*a*),

$3 \cdot 2$ sides, perimeter $6r \cdot 2 \sin (60°/2)$;

and

$3 \cdot 2^2$ sides, perimeter $6r \cdot 2^2 \sin (60°/2^2)$;

and so on, to

$3 \cdot 2^n$ sides, perimeter $6r \cdot 2^n \sin (60°/2^n)$.

It is geometrically obvious that the perimeters increase consistently as n increases.

Comparing these sequences with those of § 2 (*c*), it is clear that the two sequences here tend to a common limit $2\pi r$.

We assume as axiomatic that this limit is also the circumference of the whole circle. Further, $2\pi r$ is greater than the perimeter of any inscribed polygon of the sequence above, and less than that of any circumscribing polygon of the sequence above. Hence by calculating perimeters of such polygons we can fix upper and lower bounds for π (see below).

5. Numerical value of π.

An approximate value of π may be obtained from the area, πr^2, of a circle of radius r; if a fair-sized circle be drawn on millimetre-squared paper and the squares within be counted, one may attempt 3 or 4 significant figures; or one may fix upper and lower bounds from the number of squares within the circle and the number cut by the circumference.

Upper and lower bounds to the value of π were calculated from the perimeters of circumscribed and inscribed regular polygons, as explained in § 4 (by a method equivalent to calculating tan 30°, tan 15°, tan $7\frac{1}{2}$, ... in succession by $(1 - \cos 2A)/(1 + \cos 2A) = \tan^2 A$, and sin 30°, sin 15°, sin $7\frac{1}{2}$°, ... in succession by $1 - \cos 2A = 2 \sin^2 A$) as follows:

Archimedes,	$6 \cdot 2^4$ sides,	$3{\cdot}1428 > \pi > 3{\cdot}1408$;
Arya Bhata,	$6 \cdot 2^6$ sides,	3·1416 from both sequences;
Vieta (1579),	$6 \cdot 2^{16}$ sides,	$3{\cdot}1415926537 > \pi > 3{\cdot}1415926535$.

Vieta's value is probably sufficiently accurate for any practical application, but the calculation was carried forward by Romanus to 16 significant figures, by van Ceulen to 21, by Snell to 35, and by Grienberger (1630) to 40 figures. Later computers used infinite series and, if only as a warning as to misdirected effort, Shanks' 707 places (1853) may be mentioned.* F. J. T.

* But concerning the accuracy of Shanks' later figures, see D. F. Ferguson, Note 1889, *Gazette* May, 1946.

1510. It is a curious thing this night perspective. Many motorists believe that their cars run better after dark, and I remember, as a boy, wondering why it was that I seemed able to free-wheel uphill on my bicycle at night. The fact is that distances seem less in the dark whereas the height of objects remains the same, and therefore the angle of approach to any particular spot seems very much steeper than it really is.—Robert Gibbings, *Sweet Thames Run Softly*, p. 18. [Per Mr. F. W. Kellaway.]

A THEOREM IN UNIFORM CONVERGENCE.

By R. L. Goodstein.

One of the most striking results in the theory of uniform convergence asserts that for a *monotonic* sequence of continuous functions, continuity of the limit function is a necessary and sufficient condition for uniform convergence. Much has been written about this theorem, but the fact that the proofs given in such works as Bromwich's *Infinite Series* and De La Vallée Poussin's *Cours d'Analyse* are incomplete, seems to justify the hope that this account may be of some assistance to students and teachers of convergence.

Our theorem is that, if for each value of $n \geqslant 0$, and at each point x of (a, b), $f_n(x)$ is continuous, $f_{n+1}(x) > f_n(x)$, and $f_n(x) \to f(x)$, then the continuity of $f(x)$ is a necessary and sufficient condition for the uniform convergence of $f_n(x)$. Write $\phi_n(x) = f(x) - f_n(x)$; we have to prove that if $\phi_n(x)$ steadily decreases to zero, then $\phi_n(x)$ converges uniformly if, and only if, each $\phi_n(x)$ is continuous.

One-half of the theorem is trivial, for *if* $\phi_n(x) \to 0$ uniformly in (a, b), then $\phi_n(x) < 1/3k$, $n \geqslant n(k)$, for all x in (a, b), and since $f_{n(k)}(x)$ is continuous we can find $\nu(k)$ so that $| f_{n(k)}(X) - f_{n(k)}(x) | < 1/3k$ for any x, X in (a, b) such that $| X - x | < 1/\nu(k)$. Hence

$$\begin{aligned} | f(X) - f(x) | &= | \phi_{n(k)}(X) - \phi_{n(k)}(x) + f_{n(k)}(X) - f_{n(k)}(x) | \\ &\leqslant | \phi_{n(k)}(X) | + | \phi_{n(k)}(x) | + | f_{n(k)}(X) - f_{n(k)}(x) | < 1/k, \end{aligned}$$

provided $| X - x | < 1/\nu(k)$, so that $f(x)$ is continuous in (a, b).

It remains to show that if $f(x)$ is continuous in (a, b), that is, if each $\phi_n(x)$ is continuous in (a, b), then $\phi_n(x) \to 0$ uniformly in (a, b). We shall consider three widely different methods of proof.

First proof.

Let $\psi_n(x)$ be any function (not necessarily positive or monotonic) such that $\psi_n(x) \to 0$ at each point x of (a, b).

If $\psi_n(x)$ does *not* converge uniformly in (a, b), then we can find an integer $q > 0$, a sequence of points $x(r)$, all in (a, b), and a corresponding strictly increasing sequence of integers $n(r)$, such that

$$| \psi_{n(r)}(x(r)) | \geqslant 1/q$$

(this is simply a detailed formulation of the negative of uniform convergence).

By the Bolzano-Weierstrass theorem, the infinite sequence of points $x(r)$ in the interval (a, b) has at least one limit point ξ in (a, b), with the property that any interval i containing ξ in its interior (if ξ lies in the open interval $[a, b]$, or having ξ for an end-point if ξ is an end-point of (a, b)) contains infinitely many points of the sequence $x(r)$; denote these points by $x_i(r)$ and the corresponding values of n by $i(r)$, arranged so that $i(r+1) > i(r)$, then

$$| \psi_{i(r)}(x_i(r)) | \geqslant 1/q.$$

Thus far we have proceeded without appeal to the special properties of $\phi_n(x)$. Now apply the foregoing result to $\phi_n(x)$ and we have

$$\phi_{i(r)}(x_i(r)) \geqslant 1/q,$$

whence, since $\phi_n(x)$ is steadily decreasing,

$$\phi_m(x_i(r)) \geqslant 1/q, \quad \text{for all } m \leqslant i(r);$$

but $i(r)$ is strictly monotonic increasing, and so given any n we can find r_n such that $i(r_n) > n$ and therefore, *for all values of* n,

$$\phi_n(x_i(r_n)) \geqslant 1/q.$$

Since $\phi_n(\xi) \to 0$, we can choose $N(q)$ so that

$$\phi_n(\xi) < 1/2q, \quad n \geqslant N(q).$$

Hence, for $n \geqslant N(q)$, $\quad \phi_n(x_i(r_n)) - \phi_n(\xi) > 1/2q,$

that is, in any interval i containing ξ we can find a point $x = x_i(r_n)$ such that $\phi_n(x)$ exceeds $\phi_n(\xi)$ by more than $1/2q$, which proves that, for $n \geqslant N(q)$, $\phi_n(x)$ is discontinuous at $x = \xi$, and therefore $f(x) = f_n(x) + \phi_n(x)$ is discontinuous at ξ (for $f_n(x)$ is continuous).

Thus if $\phi_n(x)$ is *not* uniformly convergent in (a, b), then $f(x)$ is discontinuous in (a, b), which completes the proof.

(The condition that $\phi_n(x)$ is steadily decreasing is needed to pass from the inequality $\phi_{i(r)}(x_i(r)) \geqslant 1/q$ to $\phi_n(x_i(r_n)) \geqslant 1/q$; from $\phi_{i(r)}(x_i(r)) \geqslant 1/q$ we can deduce $| \phi_{i(r)}(x_i(r)) - \phi_{i(r)}(\xi) | > 1/2q$ for sufficiently great values of r, but this does not prove $\phi_{i(r)}(x)$ is discontinuous at ξ since the function $\phi_{i(r)}(x)$ itself changes as we change the interval i containing the point ξ.)

Second proof.

The following proof is shorter but reveals rather less of the intrinsic character of the theorem.

We observe that if $\phi_n(x)$ is continuous in (a, b), then $\phi_n(x)$ has an exact upper bound $\mu(n)$ in (a, b) which is attained at some point $x(n)$ in (a, b). By the Bolzano-Weierstrass theorem the infinite sequence of points $x(n)$ has at least one limit point ξ, say, in (a, b), such that any interval i containing ξ contains infinitely many points of the sequence $x(n)$, say $x(i(r))$, where $i(r)$ is steadily increasing.

Since $\phi_n(\xi) \to 0$, we can find c, depending on k, such that $\phi_n(\xi) < 1/2k$, $n \geqslant c$, and since $\phi_c(x)$ is continuous we can find an interval i, depending upon k, such that

$$| \phi_c(\xi) - \phi_c(x) | < 1/2k \quad \text{for all } x \text{ in } i.$$

Therefore $\quad \phi_c(x) < 1/k$ for all x in i.

But in the interval i lies the points $x(i(r))$ where $i(r+1) > i(r)$; choose ν so that $i(\nu) \geqslant c$, then $\phi_c(x(i(\nu))) < 1/k$, and since $\phi_n(x)$ steadily decreases (observe that this is the first point in the proof where the condition that $\phi_n(x)$ decreases is needed), therefore

$$\phi_{i(\nu)}(x(i(\nu))) < 1/k, \quad \text{that is,} \quad \mu(i(\nu)) < 1/k.$$

However, $\qquad \mu(n+1) = \phi_{n+1}(x(n+1)) < \phi_n(x(n+1)) \leqslant \mu(n),$

so that $\mu(n)$ is steadily decreasing, and therefore $\mu(n) < 1/k$, $n \geqslant i(\nu)$. Hence for all x in (a, b), $\phi_n(x) < 1/k$, $n \geqslant i(\nu)$, where ν depends only upon k, so that $\phi_n(x)$ converges uniformly to zero.

Third proof.

This proof is the most important of the three in that it proceeds by establishing a general theorem of which our result is a simple corollary. It depends, however, upon deeper notions in analysis than either of the foregoing proofs, and is suitable only for advanced students.

We start by proving a very simple necessary and sufficient condition for the continuity of the limit of any convergent sequence of continuous functions.

Theorem. If $f_n(x)$ is continuous and if $\phi_n(x) = f(x) - f_n(x) \to 0$, in (a, b), then a necessary and sufficient condition for the continuity of $f(x)$ in (a, b) is that corresponding to each integer k, and each point x in (a, b), we can find an integer $\nu = \nu(k, x)$, and an interval $\mathfrak{I}(k, x)$ such that

$$| \phi_\nu(y) | < 1/k \quad \text{for all } y \text{ in } \mathfrak{I}(k, x).$$

The condition is necessary, for if $f(x)$ is continuous, then $\phi_n(x)$ is continuous for any n, and so, since $\phi_n(x) \to 0$, we can determine $\nu = \nu(k, x)$ and an interval $\Im(k, x)$ containing x, such that

$$| \phi_n(x) | < 1/2k, \quad n \geqslant \nu(k, x)$$

and
$$| \phi_\nu(x) - \phi_\nu(y) | < 1/2k \quad \text{for any } y \text{ in } \Im(k, x),$$

whence
$$| \phi_\nu(y) | < 1/k \quad \text{for any } y \text{ in } \Im(k, x).$$

The condition is sufficient for, writing $\nu = \nu(k, x)$, since $f_\nu(u)$ is continuous at all points u of (a, b) (for any x in (a, b)), we can determine an interval $i(k, x)$ such that

$$| f_\nu(x) - f_\nu(y) | < 1/k \quad \text{for all } y \text{ in } i(k, x),$$

and therefore

$$| f(x) - f(y) | \leqslant | f_\nu(x) - f_\nu(y) | + | \phi_\nu(x) | + | \phi_\nu(y) | < 3/k$$

for all y in the common part of the intervals $i(k, x)$, $\Im(k, x)$, which proves that $f(x)$ is continuous at all points x of (a, b), and completes the proof of the general theorem.

The deduction of our initial theorem on uniform convergence requires an application of the Heine-Borel theorem.

We notice first that if $\phi_n(x)$ is positive and steadily decreasing, then

$$| \phi_\nu(y) | < 1/k$$

implies $\phi_n(y) < 1/k$ for all $n \geqslant \nu$ and all y in $\Im(k, x)$.

Accordingly, *for a monotonic increasing sequence*, a necessary condition for continuity of the limit function is that, corresponding to each point x of (a, b) there is an integer $\nu(k, x)$ and an interval $\Im(k, x)$, containing x, such that

$$\phi_n(y) < 1/k \quad \text{for all } n \geqslant \nu(k, x) \text{ and all } y \text{ in } \Im(k, x).$$

Keeping k fixed, by the Heine-Borel theorem we can select from the totality of intervals $\Im(k, x)$ a *finite* number which cover all points of the closed interval (a, b). Let these intervals be $\Im(k, x_1), \Im(k, x_2), \ldots \Im(k, x_p)$; if x is any point in (a, b), then x lies in $\Im(k, x_r)$ for a certain r not greater than p, and therefore

$$\phi_n(x) < 1/k \quad \text{for } n \geqslant \nu(k, x_r).$$

Hence if $n(k)$ is the greatest of the numbers $\nu(k, x_m)$, $m = 1, 2, \ldots, p$, then $\phi_n(x) < 1/k$ for $n \geqslant n(k)$ and all x in (a, b), where $n(k)$ is independent of x, and so $\phi_n(x)$ is *uniformly* convergent to zero, and the theorem is proved.

We conclude by noting two further important theorems on the continuity of the limit of a sequence of functions.

Definition. A sequence of functions $f_n(x)$ is continuous in (a, b) *uniformly in* n if we can determine a positive integer $\nu(k)$, not depending on n, such that

$$| f_n(X) - f_n(x) | < 1/k$$

for any x, X in (a, b) satisfying $| X - x | < 1/\nu(k)$.

Theorem A. If $f_n(x) \to f(x)$ at all points of (a, b), and if $f_n(x)$ is continuous in (a, b) uniformly in n, then $f(x)$ is continuous.

For $| f(x) - f_n(x) | < 1/k$, $n \geqslant n(k, x)$, and so

$$| f(X) - f(x) | \leqslant | f(X) - f_n(X) | + | f_n(x) - f(x) | + | f_n(X) - f_n(x) |$$
$$< 3/k, \quad n > \text{Max}\{n(k, x), n(k, X)\}, \quad | X - x | < 1/\nu(k).$$

Definition. A sequence of functions $f_n(x)$ is continuous in (a, b) *relative to* n if we can determine positive integers $v(k)$, $N(k, x, y)$ such that $|f_n(X) - f_n(x)| < 1/k$ for any x, X in (a, b) satisfying $|X - x)| < 1/v(k, x)$, and for $n \geqslant N(k, x, X)$.

Theorem B. If $f_n(x) \to f(x)$ at all points of (a, b), then the continuity of $f_n(x)$ in (a, b) relative to n, is a necessary and sufficient condition for the continuity of $f(x)$ in (a, b).

For

$$f(X) - f(x) = \{f(X) - f_n(X)\} + \{f_n(x) - f(x)\} + \{f_n(X) - f_n(x)\}, \quad \ldots\ldots(1)$$

and $|f(X) - f_n(X)| < 1/k$, $n \geqslant n(k, X)$, $|f(x) - f_n(x)| < 1/k$, $n \geqslant n(k, x)$.

Hence, if $f(x)$ is continuous, we can determine $v(k, x)$ so that

$$|f(X) - f(x)| < 1/k, \ |X - x| < 1/v(k, x),$$

and therefore

$$|f_n(X) - f_n(x)| < 3/k \text{ for } |X - x| < 1/v(k, x), \quad n \geqslant \text{Max}\{n(k, X), n(k, x)\},$$

so that $f_n(x)$ is continuous relative to n; and, conversely, if $f_n(x)$ is continuous relative to n, then from (1), for $n \geqslant \text{Max}\{n(k, x), n(k, X), N(k, x, X)\}$, we have $|f(X) - f(x)| < 3/k$ for $|X - x| < 1/v(k, x)$, which proves that $f(x)$ is continuous. R. L. G.

ANNUAL GENERAL MEETING, 1947.

THE Annual General Meeting, 1947, of the Mathematical Association, will be held on April 9th and 10th at the Polytechnic, Regent Street, London. Details of the programme will be circulated later, but it may be stated now that the Presidential Address will be entitled "A Century of School Mathematics". Also included will be papers on the Theory of Numbers, on "Proof by Reductio ad Absurdum", and on aspects of Statistics and Mathematical Astronomy as related to the school course. There will also be a discussion on the Integrated Course in Secondary School Mathematics, and a session will be devoted to Visual Aids. This will include a short Paper and a considerable number of demonstrations. There will also be an exhibition of Visual Aids as well as the usual Book exhibition. Meals will be obtainable on the premises.

A further possibility is an evening function on Wednesday, April 9th, to be held from about 6.30 p.m. to 8.30 p.m. and including some entertainment and a supper. This will be arranged if a sufficient number of members will signify their expectation of their being present at such a gathering. Friends of members would also be welcome. In order that the demand for this evening function may be determined, would members hoping to attend it please send a note to the Secretary of the Programme Committee (Mr. F. W. Kellaway, Southill, Heath Lane, Hemel Hempstead, Herts.) **immediately** on receipt of the *Gazette* in which this notice appears. A card with some such wording as "I should like to attend an evening function during the annual Meeting and would probably bring x friends", would be adequate. The cost will be about 2s. per head.

It is also requested that members who have material of any sort whatever that might be suitable for the Visual Aids Exhibition (models, drawings, charts, film strips, etc.) would please communicate with Mr. I. R. Vesselo, 80 Ventnor Drive, Totteridge, London, N. 20.

The Programme Committee are most grateful for the many suggestions and offers of help that have been sent in connection with the forthcoming meeting. It has not been possible to include everything in the 1947 programme, but all ideas submitted will be borne in mind for future years. F. W. K.

SOLUTION OF CABLE PROBLEM BY LAPLACE TRANSFORM.

By N. W. McLachlan.

1. *Introduction.* Most of the tractable problems pertaining to submarine cables and electrical transmission lines have been solved. However, no result seems to have been given for the current at any point of a very long "loaded" cable with a series capacitance between it and the sending battery. By aid of a general theorem published recently in the *Mathematical Gazette*,* the required result may be obtained without difficulty.

2. *Loaded and unloaded cables.* The electrical essentials of the latter are a central copper conductor surrounded concentrically by a dielectric having low loss and a high insulation resistance. The inductance and leakance (conduction through the dielectric to earth) of the cable are assumed to be negligible in comparison with its capacitance and resistance.

About half a century ago Heaviside showed that a substantial increase in inductance would improve the shape of the signal characters at a given speed of sending, so that for a given legibility, the speed of signalling could be increased accordingly. By using modern magnetic materials of high initial permeability,† the inductance may be increased 60-fold or more. The magnetic material, in the form of fine wire or thin tape, is wound round the central copper conductor, the dielectric being outside it. A cable of this type is said to be "loaded". Many miles of it are in use throughout the world.

3. *The problem.* We consider a very long loaded cable having resistance $\mathbf{R}$, capacitance $\mathbf{C}$, and inductance $\mathbf{L}$, all reckoned for a length of one nautical mile (6080 feet), and taken to be independent of signal wave-form. The leakage $\mathbf{G}$ is assumed to be negligible. The cable is so long that, at an intermediate point x sufficiently remote from the far end (insulated from earth), the current reflected therefrom which reaches x, is small enough to be ignored. One side of a capacitance C_1 is connected to the cable at $x=0$. At time $t=0$, a battery of potential difference E_0 is applied between the free side of C_1 and earth. What is the current at any point of the cable distant x from C_1, at time $t \geqslant 0$?

4. *Laplace transform solution.* This is given by ‡

$$I(x,t) \rightleftharpoons E_0 e^{-\lambda x}/(Z_s + Z_c), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where Z_s is the impedance of C_1 (the "sending" capacitance), Z_c the characteristic or surge impedance of the cable, and λ the transmission coefficient, these expressed as functions of p. The corresponding L.T.S. (functions of p) are:

$$Z_s = (pC_1)^{-1}; \quad Z_c = \{(p\mathbf{L}+\mathbf{R})/p\mathbf{C}\}^{1/2}, \quad \lambda = \{(p\mathbf{L}+\mathbf{R})\,p\mathbf{C}\}^{1/2}. \quad \ldots\ldots (2)$$

Writing $a = \frac{1}{2}\mathbf{R}/\mathbf{L}$, $\rho = (\mathbf{C}/\mathbf{L})^{1/2}/C_1$, $v = 1/(\mathbf{LC})^{1/2}$ the velocity of propagation along the cable, and using (2) in (1), leads to the L.T. solution:

$$I(x,t) \rightleftharpoons E_0(\mathbf{C}/\mathbf{L})^{1/2}\, p e^{-\frac{x}{v}\{(p+a)^2-a^2\}^{1/2}} \Big/ [\{(p+a)^2 - a^2\}^{1/2} + \rho]. \quad \ldots\ldots (3)$$

We have now to interpret the r.h.s. of (3) in terms of t, *i.e.* invert the Laplace transform.

5. *Interpretation of* (3), § 4.

1°. We use the known theorem:

If $\quad f(t) \rightleftharpoons \phi(p)$, then $e^{at} f(t) \rightleftharpoons \left(\frac{p}{p-a}\right)\phi(p-a)$. $\quad \ldots\ldots\ldots\ldots (1)$ §

* **30, 85, 1946.** † The permeability when the magnetising force $H \to 0$.

‡ McLachlan, *Complex Variable and Operational Calculus*: (1) is the first term of (2) § 13.42. The sign $\rightleftharpoons$ means that the function of p on the r.h.s. is the L.T. of $f(t)$ on the l.h.s.

§ *loc. cit.* (3), § 8.411.

Writing $(p-a)$ for p in (3), § 4, multiplying by $p/(p-a)$, and putting $b=x/v$, we get

$$e^{at}I(x,t)/E_0(\mathbf{C}/\mathbf{L})^{1/2} = pe^{-b(p^2-a^2)^{1/2}}/[(p^2-a^2)^{1/2}+\rho]. \quad \ldots\ldots\ldots\ldots(2)$$

The r.h.s. of (2) may be expressed in the form :

$$\frac{pe^{-b(p^2-a^2)^{1/2}}}{(p^2-a^2)^{1/2}} - \frac{\rho p e^{-b(p^2-a^2)^{1/2}}}{(p^2-a^2)^{1/2}[(p^2-a^2)^{1/2}+\rho]}. \quad \ldots\ldots\ldots\ldots(3)$$

Then $$I_0[a(t^2-b^2)^{1/2}] = pe^{-b(p^2-a^2)^{1/2}}/(p^2-a^2)^{1/2}, \quad \ldots\ldots\ldots\ldots(4)^*$$

I_0 being a modified Bessel function of zero order, so we have now to interpret the second member of (3) in terms of t. To accomplish this we need another theorem.

2°. Referring to the *Mathematical Gazette, loc. cit.*, we shall employ the theorem in § 5 (c), namely :

If $f(t) = \phi(p)$, then

$$\int_b^t I_0[a(t^2-w^2)^{1/2}]f(w)\,dw = \frac{p}{p^2-a^2}\cdot\phi[(p^2-a^2)^{1/2}], \quad \ldots\ldots\ldots\ldots(5)$$

where $w \geqslant b = x/v$.

3°. Before applying (5) we must know $f(w)$. If in the third member of

$$f(w) = e^{-\rho(w-b)} = e^{-bp}p/(p+\rho), \quad \ldots\ldots\ldots\ldots(6)$$

we write $(p^2-a^2)^{1/2}$ for p, and multiply by $-\rho p/(p^2-a^2)$, the second member of (3) is reproduced. Now by aid of the former paper, it may be demonstrated that (5) is valid when $f(w)$ is the second member of (6). Hence we have shown that

$$-\rho e^{\rho x/v}\int_{x/v}^t e^{-\rho w}I_0[a(t^2-w^2)^{1/2}]\,dw \geqslant -\rho p e^{-\frac{x}{v}(p^2-a^2)^{1/2}}\Big/(p^2-a^2)^{1/2}[(p^2-a^2)^{1/2}+\rho]. \quad \ldots(7)$$

6. *Solution in terms of t.* From (2)-(4), (7), § 5, we find that the current at any point x of the cable is represented by

$$I(x,t) = E_0(\mathbf{C}/\mathbf{L})^{1/2}e^{-at}\{I_0[a(t^2-x^2/v^2)^{1/2}] - \rho e^{\rho x/v}\int_{x/v}^t e^{-\rho w}I_0[a(t^2-w^2)^{1/2}]\,dw\}. \quad (1)$$

To find the effect of replacing C_1 by a short circuit, we make it infinite. Then ρ vanishes, and the current is given by

$$I(x,t) = E_0(\mathbf{C}/\mathbf{L})^{1/2}e^{-at}I_0[a(t^2-x^2/v^2)^{1/2}], \quad \ldots\ldots\ldots\ldots(2)$$

the known result.† Moreover, the second term on the r.h.s. of (1) represents the reduction in current due to the presence of C_1, when $t > x/v$. At $t = x/v$, the integral in (1) is zero, and

$$I(x, x/v) = E_0(\mathbf{C}/\mathbf{L})^{1/2}e^{-ax/v}. \quad \ldots\ldots\ldots\ldots(3)$$

Thus C_1 has no effect on the initial value of the current, since an uncharged capacitance acts momentarily as a short circuit. At $x=0$, the current jumps to the value $E_0(\mathbf{C}/\mathbf{L})^{1/2}$, $(\mathbf{L}/\mathbf{C})^{1/2}$ having the dimensions of a resistance. The current travels along the cable at velocity $v = 1/(\mathbf{LC})^{1/2}$, with a vertical wave front (viz. the jump part), being attenuated *en route* in accordance with the factor $e^{-ax/v}$. At any point x, for $t > x/v$, the current decays asymptotically to zero with increasing time, in accordance with (1). N. W. McL.

* *loc. cit.* (8), § 8.731. † *loc. cit.* (7), § 13.33, with $b=0$, since $\mathbf{G}=0$.

MATHEMATICAL NOTES.

1936. *The orthopole.*

In his interesting paper on the orthopolar circle (*Gazette*, 1941, p. 288), F. H. V. Gulasekharam proves various theorems about the orthopole, several of which he believes to have been mentioned before. The following bibliographical notes may be of interest.

(1) The fact that the orthopolar circle of a line cuts orthogonally the pedal circle of every point on the line is known as Lemoyne's theorem, published in the *Nouvelles Annales de Mathématiques*, 1904, p. 400. This theorem, of which several proofs have been given, has been generalised by Bricard in the same periodical (1920, p. 200): in a tangential net of conics, the auxiliary circles of the conics having a focus on a given straight line cut orthogonally a fixed circle.

(2) The property already given in H. G. Forder's Note 1281 (*Gazette*, 1938, p. 83), according to which the orthopole of any of four straight lines as to the triangle formed by the three others lies on the line of orthocentres of the four triangles formed by the four lines, was already stated in my Question 2388 in the *Nouvelles Annales de Mathématiques*, 1919, p. 39.

(3) The very simple expression of the square of the radius ρ of the orthopolar circle obtained by F. H. V. Gulasekharam, or the constant power of the orthopole ϖ of a line Δ with respect to the pedal circle of a variable point on Δ, is known in a slightly different form: if θ_a, θ_b, θ_c are the angles which the sides of the triangle ABC make with Δ, M the area of the triangle and p, q, r the lengths of the perpendiculars from A, B, C on Δ, then

$$2M\cos\theta_a = ap - bq\cos C - cr\cos B, \ldots,$$

and the distance d from the circumcentre to Δ is given by

$$2Md = R(ap\cos A + bq\cos B + cr\cos C),$$

R being the circumradius. But the distance δ from ϖ to Δ is

$$2R\cos\theta_a\cos\theta_b\cos\theta_c,$$

and the expression obtained by F. H. V. Gulasekharam for ρ^2 is equivalent to the property, to be found in Gallatly's *Modern Geometry of the Triangle*, p. 51, according to which $\rho^2 = 2d\delta$.

(4) The orthopolar ellipse, locus of the orthopole ϖ when Δ passes through a given point, seems to have first been considered by Neuberg, *Bulletin de l'Académie royale de Belgique*, July-August, 1910. The properties about the axes of the ellipse have been often mentioned, and particularly in a paper published in the *Educational Review*, Madras, October-November, 1924, by A. A. Krishnaswami Ayyangar.

(5) A great number of properties of the orthopole are given in my paper in the *Tohôku Mathematical Journal*, Sendai, 1926, pp. 77-125.

R. GOORMAGHTIGH.

1937. *Extension of Simson's line. Sondat's theorem.*

1. Let Δ_1, Δ_2, Δ_3, Δ_4 be four straight lines in a plane, A_{ij} the points (Δ_i, Δ_j) and B_{23}, B_{31}, B_{12} the intersection points of the perpendiculars erected on Δ_1, Δ_2, Δ_3 at A_{14}, A_{24}, A_{34}. Mr. N. M. Gibbins (Note 1727, *Gazette*, 1944, p. 107) has given interesting properties in connection with the triangles $A_{23}A_{31}A_{12}$ and $B_{23}B_{31}B_{12}$ and proved analytically that Δ_4 bisects the join of the orthocentres of these triangles.

This property is well known as Sondat's theorem (*Intermédiaire des Mathématiciens*, 1895, p. 202). Mr. Gibbins recalls a "difficult" geometric proof he gave in the *Gazette*, 1925, p. 440; various geometric proofs have been

published in *Mathesis*, 1895 (Haerens, p. 265; Meurice, p. 266; Neuberg, p. 267), 1896 (Jerabek, p. 81), and 1925 (Witwinski, p. 371); the one given by Neuberg is very simple.

The fact that the circumcircles of the considered triangles cut orthogonally, mentioned by Mr. Gibbins, has also been proved in these articles.

2. The analytic treatment of the configuration takes an interesting form, when use is made of complex coordinates, the quadrilateral formed by the lines Δ_i being considered in connection with its inscribed three-cusped hypocycloid (H). The centre ω of (H) is the Hervey-point of the quadrilateral,* intersection of the axes of the segments joining the circumcentres O_1, O_2, O_3, O_4 of the triangles formed by the lines $(\Delta_2, \Delta_3, \Delta_4)$, $(\Delta_3, \Delta_4, \Delta_1)$, $(\Delta_4, \Delta_1, \Delta_2)$, $(\Delta_1, \Delta_2, \Delta_3)$ to the respective orthocentres H_1, H_2, H_3, H_4 of these triangles.

If, in a system of complex coordinates, the tritangent circle Γ of (H) is taken as base-circle, ω being the origin and the image in ω of one of the apses of (H) being the unit-point, and if it is noted that (H) is the envelope of a chord of Γ such that its two extremities move along Γ with velocities of opposite sign, one being twice the other, the equation to a tangent to (H), joining the points t and $-t^{-2}$ on Γ, will be

$$xt^2 - \bar{x}t = t^3 - 1,$$

$\bar{x}$ being conjugate to x.

Let now t_i be the turns corresponding to the tangent Δ_i to (H) and denote by σ_1 and σ_4 the expressions

$$\sigma_1 = t_1 + t_2 + t_3 + t_4, \qquad \sigma_4 = t_1 t_2 t_3 t_4.$$

The point A_{23} has for coordinate

$$x_{23} = t_2 + t_3 + (t_2 t_3)^{-1};$$

hence, the parametric equation to the circle $A_{23}A_{31}A_{12}$ is

$$x = t_1 + t_2 + t_3 - \tau[1 - (t_1 t_2 t_3)^{-1}].$$

The centre O_4 of that circle is $\xi_4 = \sigma_1 - t_4$ and σ_1 is the centre of the Miquel circle $O_1O_2O_3O_4$.

The orthocentre H_4 of the triangle $A_{23}A_{31}A_{12}$ has for coordinate

$$x_{23} + x_{31} + x_{12} - 2\xi_4 = (t_2 t_3)^{-1} + (t_3 t_1)^{-1} + (t_1 t_2)^{-1} = s.$$

But the point B_{23} is the image of A_{23} in O_1; hence, its coordinate is

$$t_2 + t_3 + 2t_4 - (t_2 t_3)^{-1}.$$

The circle $B_{23}B_{31}B_{12}$ has therefore as parametric equation

$$x = \sigma_1 + t_4 - \tau(1 + t_4/\sigma_4),$$

and its centre O'_4 is $\sigma_1 + t_4$. It follows that the orthocentre h_4 of $B_{23}B_{31}B_{12}$ is $2t_4 - s$, and the point t_4 bisects H_4h_4.

According to the foregoing remarks, Sondat's theorem can thus be completed:

The midpoint of the join of the orthocentres of the triangles $A_{23}A_{31}A_{12}$ and $B_{23}B_{31}B_{12}$ is the image of the circumcentre of the first triangle in the mid-point of the segment joining the Hervey-point of the quadrilateral to the centre of the Miquel circle.

* Mr. Thébault has mentioned (*Bulletin de la Société royale des Sciences de Liége*, 1944, p. 293) that the theorem given by F. R. J. Hervey (*Educational Times, Reprint*, 1891, p. 37) had already been obtained by S. Kantor (*Bulletin des Sciences mathématiques et astronomiques*, 1870, p. 138).

The parametric equations to the circles $A_{23}A_{31}A_{12}$ and $B_{23}B_{31}B_{12}$ show that both circles pass through the point $\sigma_1 - \sigma_4$ (Miquel point). The segments joining that point to O_4 and O_4' are equipollent to those joining ω to the points $\sigma_4 - t_4$ and $\sigma_4 + t_4$; hence the two considered circles cut orthogonally.

3. The preceding analytic proof suggests an interesting generalisation, believed to be new.

Let C_{23}, C_{31}, C_{12} be the intersection points of the straight lines forming with Δ_1, Δ_2, Δ_3 at A_{14}, A_{24}, A_{34} equal angles θ. Then C_{23} is on the circle O_1 and O_1C_{23} forms with O_1A_{23} an angle 2θ.

Hence, if the expression $e^{2i\theta}$ is denoted by α, the coordinate of C_{23} is

$$t_2 + t_3 + (1 - \alpha)t_4 + \alpha/t_2t_3,$$

and the circle $C_{23}C_{31}C_{12}$ may be represented by the parametric equation

$$x = \sigma_1 - \alpha t_4 - \tau(1 - \alpha t_4/\sigma_4);$$

its centre is $\sigma_1 - \alpha t_4$ and the orthocentre of the triangle $C_{23}C_{31}C_{12}$ is

$$(1 - \alpha)t_4 + \alpha s.$$

The segments joining t_4 to the orthocentres of $A_{23}A_{31}A_{12}$ and $C_{23}C_{31}C_{12}$ are equipollent to those joining ω to the points

$$s - t_4, \qquad \alpha(s - t_4).$$

Hence the following generalisation of Sondat's theorem :

If, from the points where three lines Δ_1, Δ_2, Δ_3 cut another line Δ_4, lines Δ_1', Δ_2', Δ_3' are drawn forming with Δ_1, Δ_2, Δ_3 equal angles θ, the axis of the join of the orthocentres H_4 and H_4' of the triangles $(\Delta_1, \Delta_2, \Delta_3)$ and $(\Delta_1', \Delta_2', \Delta_3')$ cuts Δ_4 at a point Q such that the angle H_4QH_4' is 2θ.

The point Q remains the same for four given lines Δ_1, Δ_2, Δ_3, Δ_4 when θ varies.

The circle $C_{23}C_{31}C_{12}$ passes through the Miquel point and the segments joining that point to the centres of the circles $A_{23}A_{31}A_{12}$ and $C_{23}C_{31}C_{12}$ are equipollent to those joining ω to the points

$$\sigma_4 - t_4, \qquad \sigma_4 - \alpha t_4.$$

The points σ_4, t_4, αt_4 being on the base circle, this shows that *the angle formed by the circumcircles of the two considered triangles is θ.*

R. GOORMAGHTIGH.

1938. *A classification of locus problems.*

One of the most important tasks of Coordinate Geometry is the solution of locus problems, although there are problems which can be solved more easily and more briefly by the methods of Plane Geometry. I have experienced that locus problems involve difficulties to a number of boys, and that a classification with hints to the solutions is in many cases a great help to them. I suggest three groups of problems and methods for their solution.

(i) The condition fulfilled by the point (ξ, η) of the locus can be expressed in the form of an equation ; all the quantities of this equation are terms in ξ and η, the coordinates of the moving point (ξ, η), and the given elements.

Example. In a semicircle, a radius OA is divided by the point P so that the ratio of distances from the nearest point of the circumference of the semicircle—PA—and from the diameter (PB) is constant (λ). Find the locus of the point P.

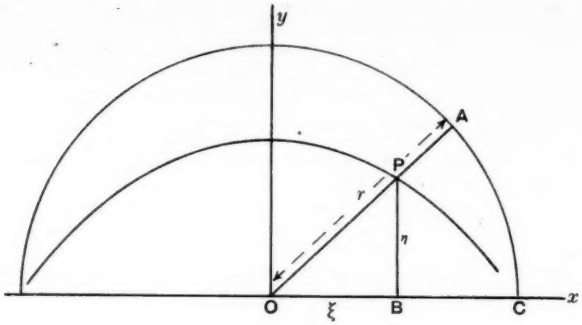

FIG. 1.

$$PA : PB = \lambda,$$

or
$$\{r - \sqrt{(\xi^2 + \eta^2)}\} : \eta = \lambda.$$

Replacing ξ by x, η by y, finally gives the equation

$$x^2 + y^2(1 - \lambda^2) + 2r\lambda y = r^2.$$

This is the equation of a conic. Which kind? That depends upon the value of λ. The diagram shows the case $\lambda = \frac{3}{5}$, when the locus is an ellipse. When $\lambda = 1$, the circle centre P and radius $PA = PB$ touches the semicircle and the diameter : the locus is a parabola. If $\lambda > 1$, the locus is a hyperbola.

(ii) The position of the point $P(\xi, \eta)$ of the locus depends upon the position of a point $C(x_0, y_0)$ on a curve with the equation $f(x, y) = 0$.

Method. Express x_0 and y_0 in terms of ξ and η and the given quantities, and substitute x_0 and y_0 in the equation $f(x, y) = 0$. The resulting equation is the required equation of the locus.

Example. Find the locus of the centroid of the triangle OBC when $OB = c$, and the vertex C moves on the parabola $y^2 = 4ax$.

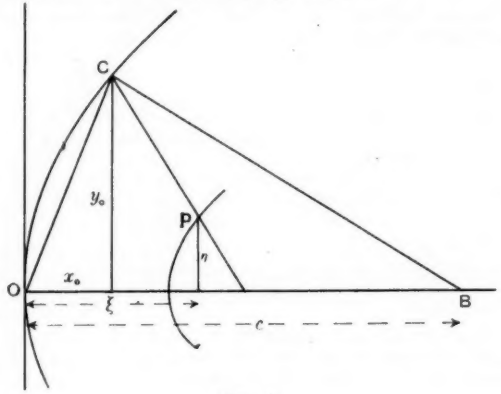

FIG. 2.

$$\begin{cases} x_0 = 3\xi - c, \\ y_0 = 3\eta, \\ y_0^2 = 4ax_0. \end{cases}$$

Eliminating x_0 and y_0 gives $3y^2 = 4a(x - \frac{1}{3}c)$.

(iii) *Method.* Introduce one auxiliary variable (or more) upon which the position of the moving point (ξ, η) of the locus depends. Express ξ and η in terms of the auxiliary variable and the given quantities. Eliminate the auxiliary quantity from the two equations for ξ and η; the resulting equation is the equation of the locus.

Example. In the triangle ABC the altitude $CO = h$, $AO = p$ and $OB = q$. Rectangles are inscribed in the triangle. One side of each rectangle lies on AB. Find the equation of the locus of the point of intersection (P) of the diagonals of the rectangle.

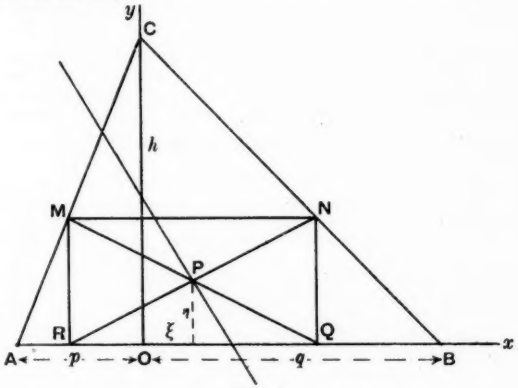

FIG. 3.

The position of the point $P(\xi, \eta)$ depends on the height (μ) of the rectangle; let μ be the auxiliary quantity. We shall get first four equations from which result two, one equation for ξ, and one for η.

(1) $\eta = \frac{1}{2}\mu$,

(2) $\xi + p = AR + \frac{1}{2}MN$,

(3) $AR : p = \mu : h$,

(4) $MN : (p + q) = (h - \mu) : h$;

or

(1) $\eta = \frac{1}{2}\mu$,

(2) $\xi + p = \dfrac{p}{h} \times \mu + \dfrac{(p+q)(h-\mu)}{2h}$.

By eliminating μ: $\quad 2h\xi + 2\eta(q - p) = h(q - p)$.

Replacing ξ by x, η by y: $\quad \dfrac{x}{\frac{1}{2}(q-p)} + \dfrac{y}{\frac{1}{2}h} = 1$.

This method will be applied, sometimes in combination with another method, especially when, for instance, a conic is given in its parametric representation. In this case the parametric quantity as the auxiliary quantity is to be eliminated in order to get the equation of the locus in x and y.

S. WEIKERSHEIMER.

1939. *On the Simson lines.*

In Note 1508 (*Gazette*, 1941, p. 58) F. H. V. Gulasekharam proves that, *if ABC, DEF be two triangles inscribed in a circle such that the Simson lines of D, E, F, with respect to ABC, meet in a point W, then the Simson lines of A, B, C, with respect to DEF, meet in the same point W, which is the mid-point of the line joining the orthocentres of the two triangles.*

The property has been proved before in the following generalised form and by purely geometrical treatment in *Mathesis*, Question 1575, proposed by S. Kantor (1906, p. 144) and solved by M. Absolonne (1907, p. 23):

If any two triangles ABC, DEF are inscribed in the same circle, the Simson lines of D, E, F, with respect to ABC, and the Simson lines of A, B, C, with respect to DEF, form two triangles inscribed in the same circle having as centre the mid-point of the segment joining the orthocentres of the two given triangles.

R. GOORMAGHTIGH.

1940. *On $\sqrt{a}$.*

It is with great diffidence that I venture to question any pronouncement of Mr. Robson, but I am not happy about what appears to be an innovation in his article on "Sign Conventions and Curvature" (*Gazette*, Feb. 1946, p. 3), in which he says: "It is perhaps worth while ... to be clear about $\sqrt{a}$. It should be restricted to the positive value of the square root, and $a^{\frac{1}{2}}$ can be used to denote $\pm\sqrt{a}$."

For more years than I care to remember I have been teaching that $a^{p/q}$ means "the real positive q-th root of a^p". Granted that in certain surdic equations such as

$$x+\sqrt{(x+22)}-2\sqrt{(x+13)}=0$$

it is usual to understand that the square roots are to be taken with the actual signs that precede them, I feel nevertheless that Mr. Robson's suggestion with regard to the use of the fractional index is going to land us in difficulties. How, for example, is he going to deal with the expansion of $(1+x)^n$ when n is a fraction?; for there is no ambiguity about the sign of the series.

Mr. Robson's convention would seem to lead to paradoxes such as:

$$x^2=x^{4/2}=\pm\sqrt{x^4}=\pm x^2.$$

I would also refer to Chrystal, *Algebra*, II (1906), pp. 188, 190, and elsewhere in the volume.

B. A. SWINDEN.

1941. *Why worry about method?*

(From a script.) Find the largest angle in the triangle whose sides are $a=4$, $b=7$, $c=9$.

$$\sin C=9/7=1{\cdot}2857.$$

$$1=\sin 90^\circ,$$

$$\cdot 2857=\sin 16^\circ\,36'.$$

Hence $$C=90^\circ+16^\circ\,36'=106^\circ\,36'.$$

B. A. SWINDEN.

EUREKA.

MANY of our members will remember with pleasure the lively *Eureka*, journal of the Cambridge Archimedeans, and will be happy to learn that after a war-time gap publication is to be resumed; the new number is expected to be ready in March. The editor is A. M. Macbeath, Clare College, Cambridge.

MATHEMATICAL ASSOCIATION TEACHING COMMITTEE.

REPORT PRESENTED TO COUNCIL 26TH OCTOBER, 1946.

THE newly constituted full Teaching Committee held a meeting on the day following the A.G.M. in April. W. J. Langford was elected Chairman and C. T. Daltry, Honorary Secretary. The Committee recorded its warm appreciation of the work of the retiring Chairman, A. Robson, in his service to the Committee and the Association.

In addition to a Standing sub-Committee, 8 other sub-Committees were appointed and terms of reference agreed for them all. The meeting also considered the second draft of Part I of the Trigonometry Report. General agreement was expressed with the form of this draft, but even so considerable constructive criticism was forthcoming. It was finally decided to refer back the Report once again to the new sub-Committee with a request that a third draft be prepared.

(*a*) *Trigonometry Sub-Committee.*

The sub-Committee has now held 3 meetings (June, August and September). The new draft is being written by Mr. Combridge and progress is sound and satisfactory. It seems likely that another 3 meetings will complete the Report and the Teaching Committee will be called to consider it as soon as it is ready. Paper will be available to print this new Report as soon as it is required. The sub-Committee (Chairman, C. O. Tuckey, and Secretary, J. C. Manisty) now consists of 10 members, 4 of them co-opted.

(*b*) *Technical Sub-Committee.*

This sub-Committee has held 2 meetings (May and September). Consideration has been given to a report on the work of the sub-Committee before 1939 and plans laid for reports on (i) the training of teachers of mathematics in technical schools and (ii) the mathematics for technical secondary schools. The first report is now well ahead and will be ready for the Teaching Committee at its next meeting. Some progress has been made with the section of the syllabus report relating to children aged 11 to 13. The sub-Committee (Chairman, F. W. Kellaway, and Secretary, H. V. Lowry) consists of 12 members, 5 of them co-opted.

(*c*) *Visual Aids Sub-Committee.*

The sub-Committee has held 2 meetings (June and September) and is engaged in preparing an exhibition for the A.G.M. in April, 1947. There will be a display of static material (contributions from members of the Association will be welcomed), and an exhibition of films followed by a discussion. The Chairman has lectured to the Mathematics Panel at the Ministry of Education. Considerable assistance is being given by Government Departments and other institutions. The Science Museum has promised a loan collection for the A.G.M.; the Ministry of Education has promised to produce some films if the Association will submit a definite proposal; the Board of Trade will assist in securing access to German mathematical models if the Ministry of Education will approve. The sub-Committee (Chairman, I. R. Vesselo, and Secretary, D. Pedoe) consists of 8 members, 2 of them co-opted.

(*d*) *Sixth Form Sub-Committee.*

This sub-Committee has met once (June) and on this occasion the whole range of the work of the sub-Committee before 1939 was reviewed and plans laid for renewed activity. There were several pre-war manuscripts available and those on Geometry and Calculus are now in circulation. The question of linking the mathematical work in the Schools with the requirements of University Departments was discussed and a preliminary recommendation made that steps be taken to ascertain the views of the Universities on the Syllabus

proposed by the Cambridge Advisory Committee. A manuscript on this topic is also circulating. The sub-Committee consists of 13 members, 3 of them co-opted.

(*e*) *Preparatory Schools Sub-Committee.*

A meeting was held in July and a draft syllabus has been drawn up at the request of the Committee of the Headmasters' Conference.

(*f*) *Modern Schools and Primary Schools Sub-Committees.*

A joint meeting was held in July but the attendance was very small; only preliminary discussion has been possible and progress is hindered by the lack of members able to speak with authority and experience in the work of these schools.

(*g*) *Examinations.*

There is nothing to report from this sub-Committee, which has not yet met.

Two other matters of general policy must be mentioned. The sale of the Association Reports is now sensational, in fact all of them are now either out of print or on the verge of being so. Supplies of Algebra, Arithmetic and Second Geometry are exhausted and reprints of 1000 copies of each have been placed. First Geometry, which had a stock of 188 in July, was down to 65 in September and a further 1000 are on order. Algebra is actually now printing and it is expected that all four orders will be completed within the course of November. Mechanics was 131 in July and is now 62 so that this will clearly demand reprinting before long. The whole position is gratifying and shows clearly the value of this aspect of the work of the Association.

Lastly, the Council should be made aware of the fact that other reports on the teaching of mathematics are being produced, and it is disturbing to find that they are coming from bodies whose function is mainly that of directing educational policy or organising professional unity. There are obvious dangers in an unlimited extension of this practice. The County Borough of Blackpool produced a report during the war for the guidance of teachers in its service, and the I.A.A.M. has just set up a Committee to deal with a new Report on Mathematics. The Teaching Committee has been in correspondence with those responsible for the I.A.A.M. Committee and has offered help, information and advice as well as suggesting that the Association would be glad to nominate representative members to serve on the new Committee. So far there has been no response to this suggestion, though the offer of assistance has been welcomed, and it is the intention of the Teaching Committee to give all possible cooperation.

W. J. LANGFORD,
Chairman—Teaching Committee.

CORRESPONDENCE.

ON SOME TEACHING SUGGESTIONS

To the Editor of the *Mathematical Gazette.*

DEAR SIR,—I am surprised that Messrs. Durell and Robson should suggest in *M.G.* 290, the *complete* disuse of the word "isosceles". Note 1867 is unconvincing. If difficulty of spelling is a satisfactory reason then "equilateral", "hypotenuse" and "symmetry" should also be banned. Spelling devices are the proper solution, *e.g.*: On a base-line *SSEE* draw two equal isosceles triangles, *SOS, ELE*. Utter the incantation, "I can spell", and then write the three initial letters in place in the figure.

The result is magical: the pupil is *spell-bound*!

I regard with mixed feelings the modern enunciation quoted in Note 1867. It is a trite, standardised statement, lacking the rhythm, verbal conciseness, smooth flow of words, and (via " isosceles ") the linguistic and historic interest of the old version, things which influence some pupils subtly, help their memory and lend interest to a lesson. True, its " if " and " then " are clearly differentiated and stated, but the pupil is thus deprived of the training involved in analysing the old version and setting out these aspects himself, while the teacher loses an interesting and informing discussion with his class. This is no " trivial matter ". It illustrates the great danger that, in simplifying and standardising Geometry for weaker pupils, the subject may lose much of its educative character and become merely a training in doing geometrical exercises, as Arithmetic sometimes degenerates into " doing sums ". It is to be hoped that enunciations like the old version, and words like " isosceles ", will always be retained for setting out some *riders* in general terms. Apart from their educative value, however, they both enable an author or examiner to vary the form and phraseology of his riders. In Ex. XX of Durell's *Elementary Geometry*, three of the first seven riders are enunciated in general terms, and five of them contain the word " isosceles ". Has Mr. Durell considered fully this aspect of the matter ? It seems to furnish a reply to Mr. Garreau's query (1st instance). Thus, some pupils, encountering the phrase " a triangle ABC in which $AB = AC$ ", would first image a scalene triangle ; a stupid pupil might even draw one. The author prevents this waste of energy by inserting " isosceles ".

The verbal perambulation necessary in setting a rider on " a parallelogram that is not a rectangle " is a pointed instance of the need for a general term, or name. Such names are essential for clarifying and fixing geometrical ideas in young minds, and for some time I have been cogitating how to deal with the classification of parallelograms for primary school pupils, in the absence of this desirable term. I should be much obliged if Messrs. Durell and Robson would kindly consider whether some term, *e.g.* RHOMB, to include rhombus and rhomboid, could not be introduced for general use.

As regards other points of the letter, the danger mentioned above again needs consideration. There is considerable educational value for pupils in learning, or discovering, that a mathematical term, or idea, is no longer adequate, and must be replaced, supplemented or extended. Thus, it is an illuminating experience for them, and an interesting one, to find, what is apparently forgotten in (iv), that, in " Graphs ", a straight line is called a curve, and that this word " curve " no longer means merely a curved line, but one of a " family " of lines. Such cases help to humanise the subject by showing mathematics as a growing organism, gradually improving ideas and practices of its youth, as it develops. The contribution this may make to the historical outlook of the pupil and to his social and moral development can only be referred to here. From this point of view, however, suggestion (ii) seems reasonable, for elementary pupils could appreciate the need for the convention. But not so the first alternative in (vii), *viz.* that AB should mean the whole line through A and B. It is sometimes forgotten that the symbol AB already has two meanings. It is both a descriptive and a quantitative term, *e.g.* AB is a line ; $AB = CD$. At present, in both aspects it indicates the same piece of line, but with the change the two pieces would be different. Pupils to S.C. standard would not see the need for this ambiguity. Nor do I see any reason for confronting them with it and what it involves. As to dispensing with such phrases as " AB produced ", they represent ideas, essential to elementary geometry, which need symbolic expression under any convention. To make AB indicate the infinite line would, no doubt, obviate the need of this

particular phrase, or, rather, its equivalent, but on some occasions only. We should still need its equivalent, and it would probably be a clumsier phrase, *e.g.* " the part of AB beyond B ". In fact, the ambiguity introduced into the meaning of AB would necessarily increase the complication of geometrical phraseology as a whole.

Suggestion (v) fails for similar reasons. S.C. pupils would hardly appreciate the distinction between quadrilateral and quadrangle, and certainly not the need for it. And the term " quadrangle ", meaning a figure of merely 4 points, *without angles*, is a most unhappy one to introduce to elementary pupils. I well remember my own long confusion over this term and the likeness of the two figures, when I started higher geometry. Similar objections rule out suggestion (iii). It postulates an ambiguous term, $x^{\frac{1}{2}}$, for pupils for whom definiteness is a first essential. Would they be expected to give four answers to an exercise such as $64^{\frac{1}{2}} + 36^{\frac{1}{2}}$, or, in logarithmic work, interpret 10^{-5} ambiguously? In the quadratic root formula would they be allowed to write as equally correct, $\sqrt{(b^2 - ac)}$ preceded by + or − or ±? Or would there be a further convention standardising one of these terms or forbidding their use?

To sum up, the general effect of adopting suggestions (vii), (v), (iii), would be to increase the difficulty and artificiality of mathematics for elementary pupils. The advanced pupil would lose much of the educational value mentioned above; and his master would lose in interest and effectiveness of teaching method, through anticipation of some of his subject-matter by the suggestions.

There remains one other suggestion, (vi), and this I am glad to support. The three-letter formulae for congruence seem to me to be used very satisfactorily by S.C. candidates. May I put forward for consideration this set of formulae for similarity :

AAA/SP, SP/AAA, SASP/S,

and, for the ambiguous case,

ASSP/AS, ASSP/2R.

The large capital S should be replaced by some approved sign for similarity, *e.g.* Greek sigma, which is easily written, especially the small letter, σ. In considering the ambiguous case, it should be borne in mind that such formulae should serve not merely as references for examiners, but also as aids to the pupil's memory of the content of the propositions.

Yours faithfully, R. S. WILLIAMSON.

NEED FOR A SYMBOL.

To the Editor of the *Mathematical Gazette*.

SIR,—Considering the ubiquity of the word " bisect " in elementary geometry, it is rather surprising that no generally recognised symbol is in existence for it.

By way of suggestion, " AB bisects CD " might be written " $AB \mid CD$ ", and, further, " AB bisects CD at right angles " as " $AB \top CD$ ".

Cracknell and Perrott use the vertical line for " is perpendicular to ", possibly because the better-known symbol $\perp$ is difficult to print (by analogy with the old factorial sign $\lfloor\_$). Does a new symbol seem called for, or are there too many already? Yours truly, C. C. PUCKETTE.

PROJECTIVE GEOMETRY.

To the Editor of the *Mathematical Gazette*.

DEAR SIR,—I think attention should be drawn to a logical slip in the article *Generalised Metrical Theorems* in the *Gazette*, Vol. XXX, No. 290, p. 122. By taking a real circle as his starting-point, the author necessarily confines himself to a *non-degenerate* conic, and so his final theorem (that the diagonals of a square bisect each other), though indisputable, is not proved by this method. There is, as far as I know, no way of passing from the real circle of theorem A, through the non-degenerate conic ϕ (the proof would not work if it were degenerate) of theorem B, to the line-pair AB, AC of the last result.

This step is akin to the " proof " of the theorem of Pappus which every examiner meets at intervals : the two given lines are a special case of a general conic ; project that conic into a circle and chase angles ; hence the result.

While I am writing, may I raise the whole question of projection into the circular points at infinity ? (The author of the above article very carefully avoids projection ; his wording could be taken as a model for the method which he describes.) I should very much like to know whether teachers believe that pupils really understand what they talk so glibly about. The whole idea seems to me full of difficulties, and I usually feel that the phrase " project into . . . " is used as a kind of charm, but that its user could seldom say just how he would do the projection. Are we, in fact, on ground where the school-boy should not stand ? The views of teachers should be very interesting and valuable.

Yours etc., E. A. MAXWELL.

TERMINOLOGY IN DYNAMICS.

To the Editor of the *Mathematical Gazette*.

SIR,—Although the word *inextensible* (of strings, etc.) is of frequent occurrence (see any examination paper in mechanics), its meaning is not always clear. For instance, when Mr. Lightfoot refers in a recent article in the *Gazette* * to an " inextensible string of length a " he appears to mean a string of unalterable length ; but when he states, a few lines further on, that " no real string fulfils the condition of being inextensible ", he seems to refer to another property, *viz.* that when the motion of a particle is checked by a string the particle does not rebound and the string afterwards remains taut. May I use your columns, Sir, to urge the use of a terminology that would avoid this confusion?

Two distinct physical properties are involved. First there is the deformability of a body. One says of a solid that it is rigid or non-rigid, or of a fluid that it is compressible or incompressible. Similarly, I suggest, one should say of a string that it is *extensible* or *inextensible* according as it can or can not be stretched. All these adjectives express purely geometrical conditions.

Secondly, there is the property that distinguishes a lump of rubber when it is deformed from a lump of putty, or a collision between steel balls from a collision between lead balls. The common idea behind these phenomena is that, when a deformation or impact occurs, internal forces or stresses come into play, and the property that distinguishes the rubber and the steel from the putty and the lead is that the work done by these internal forces during a cyclic deformation or during the impact is zero. To describe this property I suggest the word *elastic*. Thus a solid is elastic if the deformation-forces are conservative. A collision is elastic if the forces of interaction between the

* *Gazette*, XXX, No. 290, p. 129 (July, 1946).

colliding particles do no work, so that the total kinetic energy is conserved (*i.e.* the coefficient of restitution is unity if there are two particles). It may be noticed that this terminology is customary in the theory of atomic collisions. A collision that involves loss of kinetic energy is *inelastic*; when the loss of energy is as great as possible (coefficient of restitution zero) the collision may be called *completely inelastic*.

I illustrate the use of these terms by a few examples. Steel can often be treated as rigid and elastic, lead as rigid and completely inelastic. Putty is deformable and completely inelastic. A spring obeying Hooke's law is extensible and elastic. Mr. Lightfoot requires of his string that it should be inextensible and completely inelastic, and his comment might read "although real strings are often practically inextensible, no real string fulfils the condition that it is *completely* inelastic". I am, Yours, etc., F. C. POWELL.

COORDINATE NOTATION.

To the Editor of the *Mathematical Gazette*.

SIR,—A problem that crops up in many practical connections is that of the labelling, schedulling, and classifying of points, areas, and lines. I refer to such cases as the numbering and listing of the buildings of a camp or village, the preparation of a county valuation roll in such a way that the properties can readily be identified on the Ordnance Survey plan of the area, or the numbering of a system of roads and their identification on a road map.

A method of labelling that has been gaining in favour in recent years is that based on coordinates; and it is difficult to find a better one. But why is it not universally adopted? I am convinced that the answer is that the accepted method of representing the coordinates is at fault: each is shown separately; this allows of classification according to one or other of the coordinates: not according to both.

Consider, for instance, the point P with coordinates x 284, y 407. On Army grid maps this would be written 284407. Now this number gives no basis for classifying a system of points, but, if rewritten as

$$24,\quad 80,\quad 47,$$

a simple basis of classification is at once available: write down the points in the numerical order of their coordinates in the new forms. The point P falls in the square to the north-east of the point 24, 00, 00. And all points in that square will be scheduled together.

The next obvious reform will be to introduce negative digits as well as positive. P now takes the form

$$34,\quad \bar{2}1,\quad 4\bar{3}$$

(x $3\bar{2}4$, y $41\bar{3}$). All points commencing with the number 34 will be scheduled together in the square whose *centre* is the point 34, 00, 00, a more symmetrical arrangement. If P represents a building it might be labelled and identified locally by the last two digits only, $4\bar{3}$.

This notation could be extended to vectors and possibly to tensors of even higher order. But an objection would probably be raised because of the labour involved in separating the components when required for calculations involving them. Would it be possible to carry out these calculations in the new symbols as they stand? I have investigated this possibility and find that not only would it be possible, but in most cases of distinct advantage to do so. The following three examples will illustrate this; the notation is duodecimal—

why put new wine in old bottles? The multiplication table can be memorised in half-an-hour.

Consider two vectors A^r, B^r, referred to rectangular Cartesian coordinates and having as covariant components:

$$A_1 \quad \bar{4}\,.\,42, \qquad B_1 \quad 0\,.\,3\bar{4},$$
$$A_2 \quad 6\,.\,5\bar{4}, \qquad B_2 \quad 5\,.\,1\bar{6},$$
$$A_3 \quad 0\,.\,1\bar{1}, \qquad B_3 \quad 4\,.\,06$$

(the contravariant components will of course have the same values). The digits have purposely been selected at random: note how seldom it is necessary to carry forward.

Addition $C^r = A^r + B^r$

A^r	$\bar{4}60\,.\,4\bar{5}1,$		$24\bar{1}$
B^r	$05\bar{4}\,.\,\bar{3}10,$		$\bar{4}\bar{6}6$
C^r	$\bar{4}14\,.\,1\bar{4}1,$		$\bar{2}\bar{2}5$

The digits are added as in ordinary arithmetic but any "carry forward" is to be carried forward *three* places.

Scalar Product $N = A^rB_r = A^1B_1 + A^2B_2 + A^3B_3$

	$\bar{4}60\,.\,4\bar{5}1,$		$24\bar{1}$	
	$05\bar{4}\,.\,\bar{3}10,$		$\bar{4}\bar{6}6$	
$0\bar{2}0$	$0\bar{4}0$	$0\bar{1}4$	$04\bar{4}$	
	100	$\bar{1}60$	$05\bar{0}$	
		$1\bar{3}0$	400	
$\bar{2}$	$\bar{3}\,.$	6	$\bar{1}$	$N = \bar{2}\bar{3}\,.\,6\bar{1}$

The first digit of each trio of B^r is multiplied by all the first digits of A^r; the second by the second, and the third by the third. Products not affecting the second duodecimal place are omitted. The digits of N are got by summing all those in each three-square compartment and carrying forward if necessary.

The square of the magnitude of a vector, A^rA_r, is found in the same way.

Vector Product $C^r = e^{rst}A_sB_t$

	$\bar{4}60\,.\,4\bar{5}1,$		$24\bar{1}$	
	$05\bar{4}\,.\,\bar{3}10,$		$\bar{4}\bar{6}6$	
,20	,$\bar{3}0$	,$5\bar{5}$	,45	
,	,00	,00	,01	$A_2B_3 - A_3B_2$
,	,	,30	,00	
$\bar{1}$,0	$\bar{3}$,0	5,0	$\bar{4}$,0	
,	,	0,0	0,$\bar{3}$	$A_3B_1 - A_1B_3$
,	,	$\bar{2}$,0	0,0	
20,	$\bar{6}$0,	30,	20,	
,	0$\bar{1}$,	$\bar{4}\bar{5}$,	43,	$A_1B_2 - A_2B_1$
,	,	$2\bar{2}$,	00,	
$C^r = 212,$	$\bar{1}3\bar{4}\,.\,1\bar{3}\bar{4},$		$\bar{2}13$	

The first three lines of the product represent A_2B_3 (second columns of each trio) and A_3B_2 (third columns). The differences give the first digits of each trio of C^r in the bottom line. The second and third digits of C^r are found in the same way. The same method could be extended to find the contracted product of a vector and tensor of the second order if three lines are allowed for the latter.

Yours, etc., J. Halcro Johnston.

REVIEWS.

Report of the Consultative Committee on the Teaching of Mathematics in Blackpool. Pp. iv, 79. 1944(?). (Gazette & Herald Ltd., Blackpool)

This Report was prepared by a Committee set up, presumably, by the Education Committee of the County Borough of Blackpool. Thirty-two members served on the Committee (most of them teachers working in schools maintained by the Authority or associated with it) under the Chairmanship of the Headmaster of the Grammar School, with the Director of Education as Vice-Chairman and an Assistant in the Education Department as Secretary. A report presented unanimously by such a body must be regarded as an authoritative statement so that, although the Chairman in his introduction is careful to say " Teachers, to be efficient, must be responsible, informed and reasonably free ", and the Committee " puts forward its decisions as authoritative recommendations the adoption of which will, we believe, result in better co-ordination among the Schools and in better work by the pupils ", the reviewer holds the view that there exists a very real danger in the publication by a Local Authority of such a pamphlet on teaching methods in any subject in the curriculum of the schools under its control. A statement authorised by an employing Authority may well restrict the legitimate freedom of the teachers in its employ, or at least give rise to anxiety and misunderstanding.

The only date to be found on the Report is that under the signature of the Chairman and this places publication in 1944 or possibly 1945. Many of the errors and blemishes will no doubt already be known to those responsible for the issue of the pamphlet, but the greatest care is called for if a second edition should be contemplated. Some of the slips in type-setting are those which might be expected if, as seems likely, the printers have had little opportunity to set up mathematical matter, but even so careful proof-reading should have avoided most of these $\left(\frac{1}{5}$ for $\frac{1}{5}$; 2,420 lbs in 1 ton; $\frac{34}{100 \times C}$ for $\frac{34}{100} \times C$, etc.$\right)$.

The Report is in 4 sections: Number in Infants' Schools; Arithmetic in the Transition Class; Arithmetic in Junior Schools; Mathematics in Post-Primary Schools. There are also three appendixes (*sic*). Among the broad principles teachers will find some which are worth saying, and have, moreover, been well said, *e.g.* on elucidation of new processes (para. 39); on care with use of the sign = (para. 44); on avoiding marginal work (para. 46), but with these are to be found matters which will not find such ready acceptance, *e.g.* the use of the " Italian " method of division in the Junior School; the illustrations of cancelling in fractions; the justification of the method for the division of fractions (para. 140); " Volumes and densities should be taken with Senior Girls only in connection with the Science Syllabus ". On a different level but, in the opinion of the reviewer, none the less important are some points to which less care seems to have been given than should be expected in a document of this kind. The following are examples: (*a*) The figures on pp. 34, 35 are so badly drawn that distances which are intended to be equal increments are plainly unequal, (*b*) the word " compass " is wrongly used (p. 45) to denote the instrument for drawing a circle; it is correctly used in para. 193, (*c*) the use of tables of ingredients (para. 132) to illustrate fractions; the idea is sound, but surely there is no suet in a cottage loaf nor would 4/5 lb. butter be put with 2½ lbs. flour to make milk loaves even in " peace " time.

The first two sections dealing with the Infants' School and the Transition Class are good, though naturally the scope is limited. In the Primary School

(here called the Junior School) the task of laying sound foundations is paramount. This has been clearly recognised in the Report and much of the comment is sound. Multiplication and division by factors is not recommended and good reasons are given. The diagrammatic representation of fractions is well done and should be most helpful to teachers at all stages. Multiplication of money is by the " wholesale " method (M.A. Arithmetic Report, para. 8. 30) but is complicated by the omission of a separate line for farthings. Since the result of a multiplication sum is to be called a product and that of a division sum a quotient, why should the difference in subtraction be labelled " left " ? The actual printing of the fractions throughout the Report calls for drastic revision and again there are many actual errors, *e.g.* $\frac{2}{33}$ for $3\frac{2}{3}$ and $1\frac{11}{3}$ for $\frac{11}{3_1}$ (para. 140) ; $\frac{8}{9}$ for $\frac{9}{8}$ (para. 142). Decimals receive a sensible treatment, multiplication being by counting the decimal places and division by an integral divisor. In para. 164 the result of an approximate decimal division is given as ·5473 but there is an error in the working and the correct values lead to ·5474. At the end of this section a suggested syllabus allocated to the 4 years of the course sets out concisely the work to be done in each year with the warning that the " rigid sub-division of the syllabus into portions appropriate to distinct ' Standards ' is strongly to be deprecated ".

The final section dealing with Secondary Schools (here called post-primary) is not satisfactory and it would probably have been better to omit this altogether. The opinion that " emphasis should be placed on processes which have clear applications in the pupils' everyday life " is welcomed, but this should not restrict the algebra and geometry available to these children so severely as the Report indicates. With these pupils indeed there is a splendid chance to teach what little algebra, geometry and trigonometry they can appreciate in such a way that mathematics does appear for them as a unified study of practical value in everyday life. It is in this very work that many teachers in the new Modern Schools will be looking for help and guidance but the Report does not offer them much assistance.

The first appendix concerns the special needs of the " least able Modern School pupils " and stresses that individual methods are essential. Its principles are praiseworthy and the card system is recommended strongly, though not all will want to organise an Inter-House Mathematics Race.

The second appendix contains three special methods for dealing with the decimalisation of money. The M.A. Report (paras. 11.10 to 11.42) deals exhaustively with the problem, and two of the methods here are simplified versions of the " Three-place method " given there ; the remaining method constructs a table which may be used as a ready-reckoner.

The final appendix takes nearly nine pages discussing the method of checking multiplication and division sums by " die Probe ". This is a general theorem of which " casting out the nines " is a special case, but its use is altogether out of place at this stage in school arithmetic. Incidentally, in the proof of the general theorem " quotient " has been written for " remainder " in two places.

The production of this Report has undoubtedly called for the expenditure of much time and effort on the part of all members of the Committee and one does not doubt that it was given very willingly. All things considered it seems likely that the result of these labours in its present form is not a worthy reward, nor will it satisfy adequately the intentions which inspired it.

W. J. L.

Sur les ensembles parfaits. Par SOPHIE PICCARD. Pp. 196. 7.50 Swiss francs. 1942. Mémoires de l'Université de Neuchatel, tome XVI. (Gauthier Villars)

This book is concerned with properties of certain sets associated with generalisations of the Cantor Middle Third set. This set, the basis for so many examples and counter-examples in Analysis, can be introduced geometrically in the following way. Remove from the closed interval [01] first its middle third, the open interval $(\frac{1}{3}\ \frac{2}{3})$, then the middle thirds $(\frac{1}{9}\ \frac{2}{9})$ and $(\frac{7}{9}\ \frac{8}{9})$ of the two intervals remaining, next the middle thirds of the four intervals now remaining, and so on. The set C of points left is the Cantor set. It can be described arithmetically as the set of numbers in [01] which can be represented in the scale of 3 by using the digits 0 and 2 only (we assume that the sequence of digits $10\dot{0}$ is replaced by $02\dot{2}$). Problems about numbers and decimals with missing digits are familiar.*

The generalisations of the Cantor set are defined as follows. Let $n \geqslant 3$ be an integer and let $0 = a_0 < a_1 < \ldots < a_k$ (where $1 \leqslant k < n-1$) be integers chosen from 0, 1, 2, ... , n. Then $\mathcal{A} = \{a_0, a_1, \ldots, a_k\}_n$ is the set of non-negative real numbers whose representation in the scale of n contains only the digits $a_0, a_1, \ldots, a_k$; the set of such numbers in [01] is denoted by $A = [a_0, a_1, \ldots, a_k]_n$. In particular $[0, 2]_3$ is the Cantor set which can also be represented, for instance, as $[0, 2, 6, 8]_9$.

The associated sets can be defined for any linear set E, not necessarily an $\mathcal{A}$ or an A. They are the sets of distances,† of sums and of differences, and are denoted respectively by $D(E)$, $s(E)$, $d(E)$. The set of real numbers, each of which is the distance between at least one pair of points of E, is denoted by $D(E)$; $s(E)$ is the set of sums of the coordinates of pairs of points of E, while $d(E)$ is the corresponding set of differences. For example, we have $D(C) = [01]$, $s(C) = [02]$, $d(C) = [-11]$.

The results obtained in the study of the sets D, s, d associated with the sets $\mathcal{A}$ and A are so diverse and often so complicated that it will not be possible to describe them in any detail. The results we have chosen to quote can at most be regarded as representative of the simplest cases of the masses of information presented.

The sets $\mathcal{A}$ and A are all non-dense perfect sets of zero measure. Their sets of distances are again perfect. $D(\mathcal{A})$ can fill the whole ray $[0\ \infty)$ (as in the case when $\mathcal{A} = \{0, 2\}_3$), or it can be of zero measure (as in the case when $\mathcal{A} = \{0, 2, 3, 7\}_{10}$), or it can be of positive measure without filling $[0\ \infty)$. The last case does not occur if $n < 9$ but does if $\mathcal{A} = \{0, 3, 5, 8\}_9$, when the measure of that part of $D(\mathcal{A})$ contained in [01] is 1/2. A result of a different type which is established is that there is no set E such that $D(E) = C$.

The set $D(E)$ can clearly be represented as the sum of three sets E_1, E_2, E_3 which are the sets of numbers t such that the set E and its translation $E(t)$ (*i.e.* the set of points $x + t$ where x is in E) have in common respectively a finite set of points, a countable set of points, a more than countable set of points. The various possibilities about the number of points in these three sets when E is an A are examined. It is shown that A_1 is never empty; it must, however, contain either a single point (*e.g.* the point 1 when $A = [0, 1, 2, 4, 5, 6]_7$) or an infinity. A_2 may be empty (*e.g.* when $A = C$) but cannot consist of a finite number of points. A_3 is always more than countable. The remaining possibilities can all be realised: A_1 is countable when $A = [0, 1, 3]_4$

* See *e.g.* G. H. Hardy & E. M. Wright, *Theory of Numbers* {2}. Oxford, 1945, pp. 119–127.

† Professor Piccard has devoted another book to the study of sets of distances: it was reviewed in the *Gazette*, XXIV (1940), 302.

and is more than countable when $A = C$, while A_2 is countable when $A = [0, 2, 3]_6$ and is more than countable when $A = [0, 3, 4]_8$.

Professor Piccard's interest in the details of such theories as those of the present book and her earlier one is without a doubt unique. She uses in the present book methods which are almost always elementary. [The industrious reader may find it interesting to construct proofs of the results quoted.] The arguments required are, however, in many instances long and intricate—there are often a dozen or more special cases to be discussed—but all are carried out with the patience and care which readers of Professor Piccard's earlier book will expect. J. T.

Leçons et problèmes sur la théorie des corps déformables. Par ANDRÉ MERCIER. Pp. 156. 240 fr. 1943. (Gauthier-Villars)

This excellent little book consists of an exposition of the theory of deformable bodies, and the principal methods for the solution of problems connected with them. The work falls into three parts of which the first gives a simple account of tensors (of the second rank) treated by vector methods, their application to deformation and stress, leading to the expression of the stress tensor in the form

$$\Psi = \Psi(\nabla;\xi, \nabla;\dot{\xi}, T, \rho)$$

where ξ is the displacement vector, T the temperature, ρ the density, and (;) denotes dyadic multiplication. This result allows the author to isolate for consideration purely elastic ($\nabla;\dot{\xi}$ absent) and purely viscous ($\nabla;\xi$ absent) phenomena.

The second part contains a condensed but remarkably comprehensive survey of elasticity including aelotropy, strain energy, torsion and flexure, and wave propagation. The third and largest part is devoted to hydrodynamics; ideal and viscous fluids; formation of vortices; two-dimensional aerofoils.

The style is lucid, physical assumptions are clearly pointed out, and there are about 120 exercises, many of them being in the form of asking the reader to supply simple steps in the arguments in the position where they occur.

L. M. M.-T.

Theory of Lie groups. I. By C. CHEVALLEY. Pp. xii, 217. 20s. 1946. (Princeton University Press; Oxford University Press)

The Lie theory, as originally developed by Lie and his followers, dealt mostly with r-parameter families of transformations in n-dimensional Cartesian space, which were usually defined only for restricted values of the parameters, $t_1, \ldots, t_r$, and the coordinates, $x_1, \ldots, x_n$. In general such a family is not a group, since the product of two transformations may not exist if their parameters lie near the boundary of the region of definition in the t-space. Alternatively a transformation, T_1, may transform the region of definition in the x-space outside the region for which a second transformation, T_2, is defined. In this case also the resultant of T_1 followed by T_2 will not exist. However, the infinitesimal and algebraic theory of such a family, which satisfies the group conditions for restricted values of the parameters t and coordinates x, is the same as if the family were a group. It is the infinitesimal and algebraic theory of such a family with which the work of Lie, Killing and the early work of Cartan has mostly to do.

Today, a Lie group is usually understood to be a topological group, which is also an analytic manifold, the group multiplication being analytic. This is the point of view adopted in this book and the ideas of a topological group and of an analytic manifold in its entirety are explained before the discussion of the differential equations and constants of structure, which were the starting-point of the classical theory.

The book opens with an account of the group of non-singular linear transformations in n complex variables and its best-known sub-groups (unitary, orthogonal, etc.), including the symplectic group, which is the analogue of the orthogonal and unitary groups, with quaternions as coordinates instead of real or complex numbers.

The next two chapters are on topological groups and analytic manifolds. Some understanding of the generalities of topology is assumed but the exposition starts at an elementary level, The chapter on topological groups deals with covering spaces, covering groups, sub-groups, etc. The general results are applied to the calculation of the Poincaré groups of the classical groups considered in Chap. I. In connection with the orthogonal group there is an interesting account of the algebra of the " Clifford numbers ". In the chapter on manifolds an analytic manifold is defined from first principles and also vectors, vector distributions and systems in involution on a manifold.

Next come chapters on Lie groups and on the calculus of exterior forms (" The differential calculus of Cartan "). These chapters include, in what will probably be a novel form to most readers, the elements of the classical Lie theory (the infinitesimal transformations, left and right translations, canonical coordinates, invariant integrals on a group, etc.).

The book concludes with a chapter on the representation of compact Lie groups. This includes an account of the characters and their orthogonality relations and of the representative ring. The latter is the ring of functions defined on the group, which is generated by the coefficients of all representations. The " main theorem " states that some function in the representative ring is an arbitrarily good approximation to a given continuous function on the group. This is the key to the theorem that any compact Lie group is isomorphic to a group of matrices. The representative ring is also used in order to prove a generalisation (Tanaka's theorem) to non-Abelian (compact) groups of Pontrjagin's duality theorem for Abelian groups.

The subject-matter is presented in a thoroughly logical fashion and with a fine sense of style. Much of the elementary part is presented in novel form. While most of these innovations are admirable, and all are interesting, a few are, I think, open to criticism on the ground that standard methods are just as good, if not better. For example, the treatment of simply-connected spaces and the Poincaré group, which includes as a non-trivial theorem the fact that an interval in the real number system is simply-connected, seems to me cumbersome. Moreover there are spaces, in which not all " arbitrarily small " circuits are contractible, which are simply-connected in Chevalley's, but not in the ordinary sense. However, it is a fine and timely book. J. H. C. W.

General Mathematics. By C. V. DURELL. Vol. 1. Pp. viii, 312, xlvii. With answers, 5s. 6d.; without answers, 5s. Vol. 2. Pp. xxxi, 324, xl. With answers, 5s. 6d.; without answers, 5s. 1946. (Bell)

These volumes are a new arrangement of some of the material of Mr. Durell's well-known textbooks, with a little new material added.

They are the first two of a series of four volumes for boys and girls from 11 to 15. Each is meant to give a year's work, and the S.C. course is to be covered in the four volumes. There is to be a short supplementary volume with work on practical solid geometry, navigation, and the use of gradients, which lie outside the normal curriculum.

The series aims at a " homogeneous course which embodies modern ideas of mathematical teaching ". Further, it claims to " avoid the uneconomic duplication of material " which occurs when pupils use separate books for each subject. It is doubtful whether the former of these aims has been achieved.

Questions arise in such a treatment as to whether the order chosen is the best one, and whether the subjects really merge into one another. The order in the first volume starts as follows : Factors and easy fractions, easy decimals, first notions of Algebra, use of geometrical instruments, formulae, angles at a point, simple areas and volumes, easy unitary method, angles and parallels. It goes on with simple work in each subject, and finishes (nineteen chapters) with the A.B.C. of graphs, practice, congruence and isosceles triangles. The chapters are mainly taken from the author's Arithmetic, Algebra, or Geometry books, and hence each is concerned with one subject. The examples are plentiful, and are arranged as in the previous books. In the chapter on Unitary Method generalised work is introduced in many questions, *e.g.* " how much does he earn in 3 hr.? in 5 hr.? in t hr.? " This is a very correct introduction to Algebra, but it seems a pity that it comes after two chapters on formal Algebra. Similar work might have been done in Mensuration, but there the chapter starts with the formulae, $A = lb$, transformed at once to $l = A/b$ and $b = A/l$; whereas if the direct result only had been quoted in words the shorthand algebraic notation could have come to the pupil after numerical examples, generalised as in the former chapter. The Geometry is kept in quite separate chapters, except that the use of small letters does involve some Algebra.

In the second volume ratio and rate are well introduced, as in the author's Arithmetic. This chapter brings a section on similar triangles in between a set of examples on multiplying factors, and the set on proportion to which this leads, and hence appears as an interruption. Further, the section of similar triangles is really leading to a theoretical treatment, and has only three practical examples, and so does not fit well with the other material of the chapter. Trigonometry is introduced early, in two chapters, the first limited to the tangent. Too much time is spent on a careful reading of tables before practical examples are given, in the third set of examples in the chapter. There is much to be said for starting instead with practical problems using a set of tables giving angles in degrees only.

The Geometry in this volume has much that is hard for the pupil of age 12 to 13 ; inequalities with the theorems in full, and proofs of the concurrency of perpendicular bisectors of sides of a triangle, and of the altitudes of a triangle, a heavy introduction to the first examples on drawing circumcircles. Pythagoras' theorem, with the easy trigonometrical proof, is rightly given in the chapter on square roots, and is then applied to calculation of lengths of chords of a circle. The similar calculation of the length of tangents is not included ; in fact, although the construction of the incircle is given, tangents are not mentioned except in the title of this chapter.

Logarithms complete the second volume, and tables are given of logarithms, sine, cosine, and tangent of angles, reciprocal, square and square roots ; not of antilogarithms, and log sin, etc., which have not been used.

There are three sets of revision examples in each volume, suggesting terminal stopping-places, and good oral, computation, and revision papers at the end of each volume, and also an index.

Certain difficulties must occur with a book of this type. A child soon knows his way about an Arithmetic book, but will find it more difficult to find a given topic here. The arrangement of the books may tempt the teacher (or the L.E.A.) to have a set of vol. 1 for the first year, of vol. 2 for the second year, and so on, the child giving up his book at the end of a year. This would not be good, since the child needs the previous volumes for reference, especially in Geometry. A further temptation is for the teacher to accept the work as laid down for each year, in the order given, thus losing his own initiative, and producing a uniformity among children which does not exist

naturally, since the brighter ones work faster and would need the next volume.

An experiment has been made, on lines demanded by many teachers. Much of the material is good, but there is nothing essentially new in the treatment, and more research is needed both in the order, and in the presentation as a unified subject. K. S. S.

The World of Numbers. By HERBERT MCKAY. Pp. 198. 8s. 6d. (Cambridge University Press)

Mathematics is exciting. Such is one of Mr. McKay's contentions, and few readers of this journal could be found to disagree with him. The problem is, of course, to convince others; *The World of Numbers* contains many suitable texts for the preacher to the unconverted. Its range includes the history of the measurement of the earth, the arithmetic of rivers, the division of time, the energy of the sun and stars, the numbers that have attracted poets, and the use of symbols. Variety indeed!; and it would be a carping student who did not find much to appeal to his fancy and his imagination.

The good sense and human touch of Mr. McKay's writing can be typified by his references to summer time and numerologists. He points out that the artificiality of any basic "standard time" vitiates any "pseudo-religious objection" to clock alterations, and proceeds to discuss the old proverb: Early to bed and early to rise, make a man healthy, wealthy and wise. If there is any truth in this, says Mr. McKay, it should certainly apply to sundial time. There should thus be "an observable difference in health, wealth and sagacity between the eastern and western limits of the time zones". Is Cornwall superior in these respects to London; and what of the wide ranges in America?

As for numerology, anyone with a grasp of the simplest algebra can expose many of the tricks of the mind-reader; and while a music-hall or party audience is willing to be deceived, it is a pity that thinking people should allow themselves to waste time and energy in "proving" the evil of Hitler or forecasting the dates of great events by manipulation of figures.

Among more technical matters in this expansive book, there are chapters on "Proportions in the Stars" with a discussion involving Kepler's laws, on the determination of π, and on "The Ghost Quantity", which gives a remarkably concise account of complex numbers.

The publishers suggest that "this book is not a sequel to *Odd Numbers* . . . it is its twin brother". This is a happy analogy; and in the same spirit we may hope that ere long the twins might become triplets, for the more books of this calibre that are available, the more attractive will mathematics appear to the non-specialist. F. W. K.

INTERMÉDIAIRE DES RECHERCHES MATHÉMATIQUES.

THIS French quarterly, first issued in 1945, has the following aims:

Helping mathematical researches; giving information on all mathematical questions; promoting contacts between mathematicians and indicating specialists on various topics; pointing out unsolved mathematical problems and subjects for research, even if they come from other branches of science; collaborating in an international exchange of mathematical knowledge; and developing a "Centre de Documentation mathématique".

An exchange has been established between *Intermédiaire* and the *Gazette*. Members of the Association who wish to subscribe personally to the journal should communicate with M. Paul Belgodère, the director, at 55, rue de Varenne, Paris (7e).

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